9 Conclusion

Empirical arguments play an important role in mathematics. They can lead to conjectures about a mathematical situation, and they can provide a strong sense of conviction that a conjecture must be true. However, unless one can come up with an exhaustive argument or is working in a finite domain, they do not show that a given conjecture *must* be true. For this we need structural arguments, ie arguments based on mathematical properties. Some structural arguments (though perhaps not all) are illuminating, ie they not only show *that* a mathematical relationship is true, but *why* it is true. It is arguments of this kind that we should focus on in school as they can help develop students' understanding of mathematical ideas.*

In this report we have amassed considerable evidence to show that secondary school students can engage with mathematical structure, including students in 'low attaining' mathematics sets. The key here is that teachers give their students the opportunity to do so. In this project we were privileged to work with highly skilled teachers, who had established attractive working atmospheres in their classrooms and who had the confidence to take risks. Given all that (!), the methods they employed were often quite simple: for example, 'working in phases' (see chapter 6) so that students had the opportunity to explore a problem and to then share their ideas; or using devices like spider diagrams (concept maps) to let students reveal (and reflect on) what they already knew (see chapter 5).

Teachers tended to use numerical/algebraic tasks rather than geometric tasks, and the former often seemed to 'work' more successfully. This does not mean that number/algebra is somehow 'easier' than geometry (does such a proposition even make sense?). However, often the geometry tasks that the teachers used involved the circle theorems (see chapter 4) and this did throw up difficulties - for example to do with the nature of diagrams (whether they are specific or generic) and the fact that the circle theorems tend to be treated as a system, which requires fluent knowledge of the theorems and how they are linked.

The successful numerical tasks could be quite rich and challenging, but they often involved quite small areas of mathematics with which students were reasonably familiar - eg the nature of odd, even and consecutive numbers (see chapter 6). Thus these proof tasks involved 'local' proofs rather than proofs of theorems embedded in a complex network of axioms, undefined terms and other theorems. It should be possible to do the same in geometry, either by deferring the process of systematising, or by working in a more confined area of geometry, such as 'parallelograms' (see page 4.12), as long as the ideas are accessible but can offer challenging tasks.

We looked at some of the characteristics of common proof tasks, including the approaches that can be used to solve them. The project teachers tended to favour algebraic proofs (ie solutions involving the creation and manipulation of algebraic expressions), even, as was often the case, for tasks where an argument could be expressed perfectly well in narrative or visual form (eg using dot patterns). We would argue that these other forms of proof should also be valued, especially as students often find them easier to construe**. We observed that students often struggled with algebraic symbolisation; if therefore teachers prioritise algebra, they may be creating a barrier to looking at structure.

On the other hand, some tasks are difficult to solve without recourse to algebraic symbolisation (eg, question 8 on page 6.1) and teachers need to be aware of this when choosing them. With such tasks it is worth seeing whether there are ways of supporting the algebra, eg by finding visual representations of the various algebraic steps in their solution (see page 6.11).

Students often generate examples before (or instead of) looking for structure, particularly when 'doing' 'investigations'. This can be perfectly legitimate. However, we feel that students (and their teachers!) should be made more aware that it can be possible to see structure (or at least speculate about it) before considering examples. Many commonly used tasks are well suited to such a direct approach (though they may benefit from being re-cast, as with the *matchstick-squares* task on page 2.5). However, we also identified tasks where the structure is less transparent and where an empirical approach may well be helpful (see chapter 2).

We think students should be encouraged to engage in thought experiments (see chapter 8), which again means holding back on the search for empirical data. This is especially important with the growing use of ICT, where data can be generated very easily.

We also discussed the benefits of evaluating student responses. In chapter 7 we looked at selected responses to question A1 from the Longitudinal Proof Project tests. Such compilations can be a useful resource, and we have therefore put selected responses to items G1, G2b and A4c in the Appendices.

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* See for example Hanna, G (1989) Proofs that prove and proofs that explain, in PME 13, vol 2, pages 45 - 51.

** Of course, some forms of algebraic proof are easier to construe than others - see, for example, the algebraic expressions on page 6.16, column 2.