8 Thought experiments

Throughout this report we have stressed the importance of looking for structure, and of trying to do this from the beginning of the solution/proof process. This means one should consider holding back on the search for empirical data, at least for a while or unless one is well and truly lost or stuck.

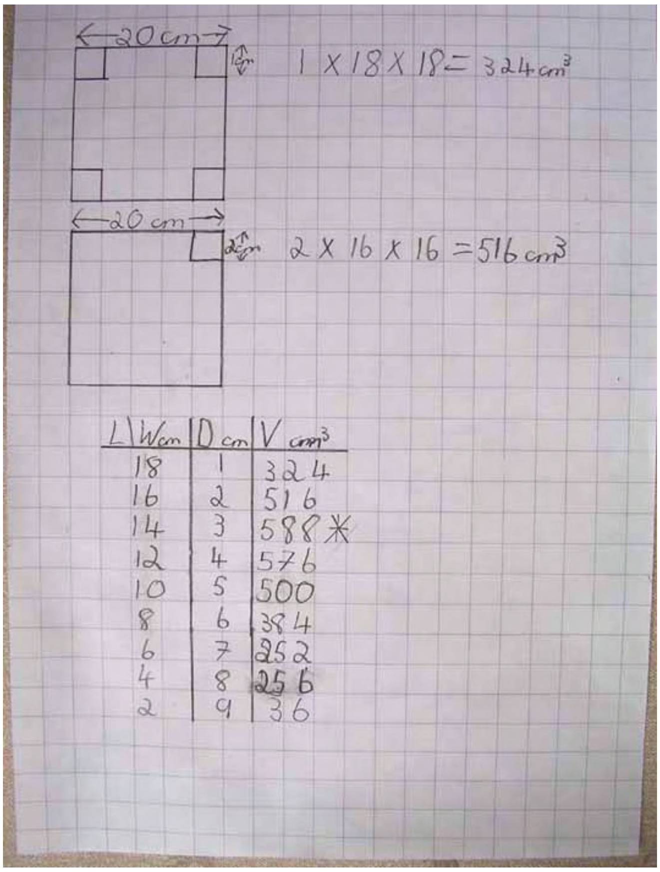
Related to this approach is the notion of 'thought experiment'. A thought experiment involves making conjectures and testing them 'in the head' rather than, or at least before, testing them in a more empirical way. It involve predicting, anticipating and monitoring, not 'generate data first, think about it later'.

Some of the tasks in the proof tests devised for the Longitudinal Proof Project lend themselves rather nicely to thought experiments (we discuss these tasks later). However, we did not find much evidence of students actually approaching them in this way. The reasons for this are unclear. Perhaps, thought experiments require rather more abstract thinking than does the generation of data. Perhaps also students are not used to working in this way, which is borne out by the fact that we rarely observed teachers promoting this approach in the current project.

Unfortunately, we did not manage to pursue this issue in any depth, although we did investigate it, none too successfully, in a small set of lessons involving *Heron's Problem* (see later).

I happened to observe one of the project teachers use the familiar task *Maxbox* with a high attaining Year 8 class. The lesson started with drawings projected on the board of several different open boxes, each made from a 20 cm by 20 cm sheet of card with identical squares cut from the four corners. The students were asked for their hunches about the volume (strictly speaking, capacity) of the boxes as larger squares are cut from the corners - does the volume increase, decrease or stay the same? This was a nice way to start but it was not pursued for long.

Instead, students started sketching the nets of specific boxes, calculating their dimensions and the resultant volumes. The example below is typical of the students' work: confident, well laid out, systematic - but perhaps with not much thought given to the significance of the results. It can be seen that the maximum volume occurs when the cut-out squares have sides 3 cm long*. However, the student has not just considered the next one or two cases (to check that 3 cm gives the maximum) but has gone all the way to the largest possible cut-out square (with sides 9 cm long). This demonstrates the student's thoroughness and diligence,



but it is not really necessary - unless there is some reason to think that there might be a second maximum.

A more extensive discussion at the beginning of the task might have allowed students to get a better feel for the task and to think more strategically. What is interesting about the task is that the volume does change and that, indeed, there is a maximum. (But how readily did the Year 8 students appreciate this? It would have been nice to have known!) One could explore this by, for example, thinking about extreme cases. Consider a box made by cutting out very small squares, (say of side 2 mm): this has a large base but is very shallow (with a height of 2 mm) and so would seem to have quite a small volume; we can make the height as close to zero as we like (by decreasing the size of the cut-out squares); this will produce larger and larger bases, but not *beliebig*** large: rather, the sides will always be less than 20 cm; thus we can make the volume as close to zero as we like. Similarly with boxes at the other extreme: we can make the base very small, with side as close to zero as we like, but with the height always less that 10 cm: so again we can make the volume as close to zero as we like. Thus there must be a maximum somewhere between the two extremes.

Of course, this does not tell us where the maximum will be, and it turns out that this is quite difficult to determine analytically (certainly for Year 8 students!).

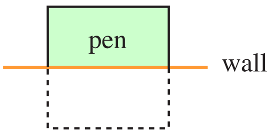
* At this stage, the class was only considering cut-out squares with sides a whole number of cm long

** Though *beliebig* is not as popular as *Blitz* or *Zeitgeist*, it is an *über*useful word whose meaning lies somewhere between 'any' and 'as you like'.

So at this stage it would be quite appropriate to look at and compare specific cases.

When I first met the Maxbox task, I thought the maximum would be reached when the box was a cube. This was using my knowledge about volume and surface area of cuboids (namely, that for a given surface area, the cuboid with the largest volume is a cube). However, this task is not about constant surface area, as pieces are cut off the given square; also it is not about closed cuboids.*

It is instructive to compare *Maxbox* and the classic *Sheep Pen* task, where the aim is to make an as-large-as-possible rectangular pen from a fixed length of fencing and a 'very long' wall. Here I initially made a similar false argument as with *Maxbox*, namely that the maximum area would be given by a square pen**. This time I was wrongly applying the fact that, for a fixed perimeter, the rectangle with the largest area is a square. However, this argument is not entirely misplaced and can lead to a correct analytic solution: it turns out the desired pen forms half a square, made from a whole square that is bisected by the wall.

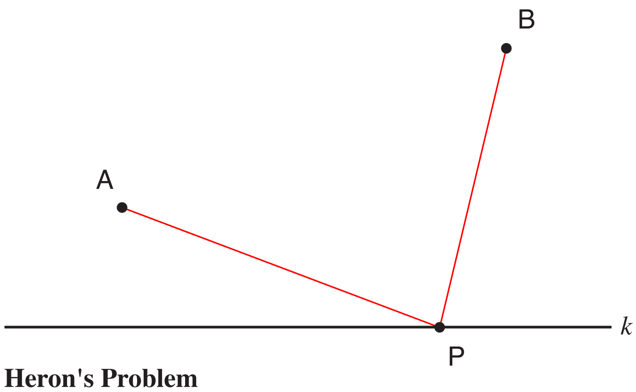


A task that has features of both *Maxbox* and *Sheep Pen* is the following:

> Take a square sheet of card. Imagine folding the sheet (with three parallel folds) to make a square-based box (without a base or top); consider the capacity of the box when it is placed on a plane surface (a table, say, or a football pitch).
>
> Imagine cutting the original sheet in half to make two identical rectangles and then join the rectangles again along their shorter sides. Make a new open square-based box (either shorter and wider or taller and thinner).
>
> Continue this process to make an ever longer/thinner rectangle (see rectangles below) and an ever shorter/wider or ever taller/thinner open box. What happens to the capacity of the box?

Experience with *Maxbox* and *Sheep Pen* may well lead to the belief that, somewhere in the middle, there will be a box with maximum capacity. It turns out that this is not the case! Using thought experiments, it is a nice challenge to explain why. (For example, imagine a box and a second box with half the first one's height; what happens to the sides of the base and hence to the area of the base?)***

Two of the project teachers tried a truly classical task, sometimes know as *Heron's Problem*****, with one of their classes. A version of the task is shown below.



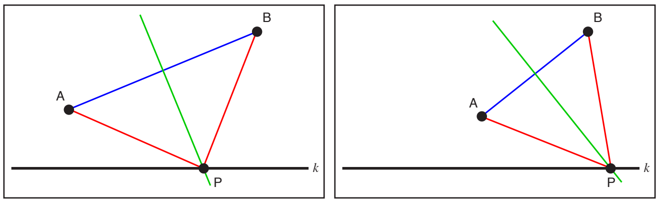
Heron's Problem

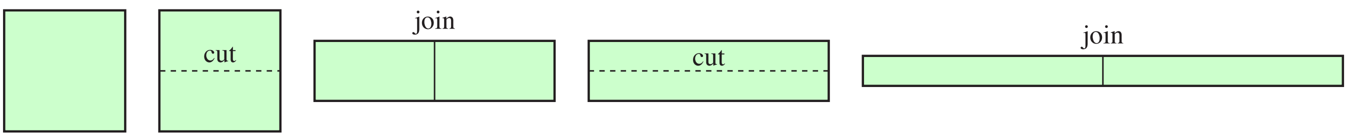
A and B are points on the same side of a line *k*.
P is a point on the line *k*.

Find the position of P where AP + PB is a minimum.

A nice feature of this task is that it readily provokes conjectures, some of which can fruitfully be addressed through thought experiments. It can also be solved in elegant, analytic ways (although it is a moot point how the necessary insight for such solutions might arise).

A common conjecture is to suppose that the desired point is equidistant from A and B, ie lies on the perpendicular bisector of AB (see below, left). This is true when A and B are the same distance from line *k*, but imagine moving A towards the line and/or towards B, into a position like the one shown below, right. The resultant point is to the right of B (ie lies outside the projection of AB onto the line *k*), which is clearly not minimal.





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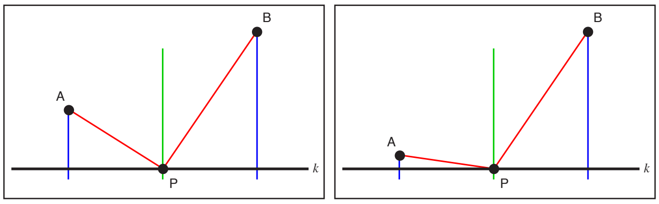
* This is a well-known kind of error, in which valid (and much prized!) mathematical knowledge is applied in an inappropriate way.

** It can be instructive to consider what happens to the area of the pen as one changes the shape from a square, to a pen that is slightly taller or slightly wider - which direction of change increases the area, and why? By considering beliebig small changes, we have here the beginnings of calculus!

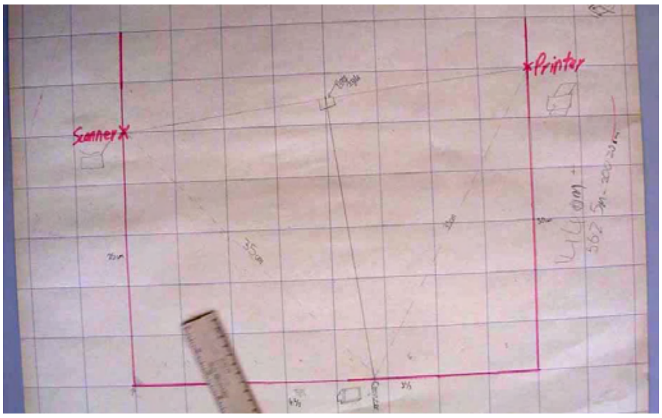
*** We came across this task (or rather a variant involving cylinders) in an article by Johnson (2006) in *Mathematics Teaching*, 198. Her approach is very different, with an emphasis on model making and on calculating using spreadsheets.

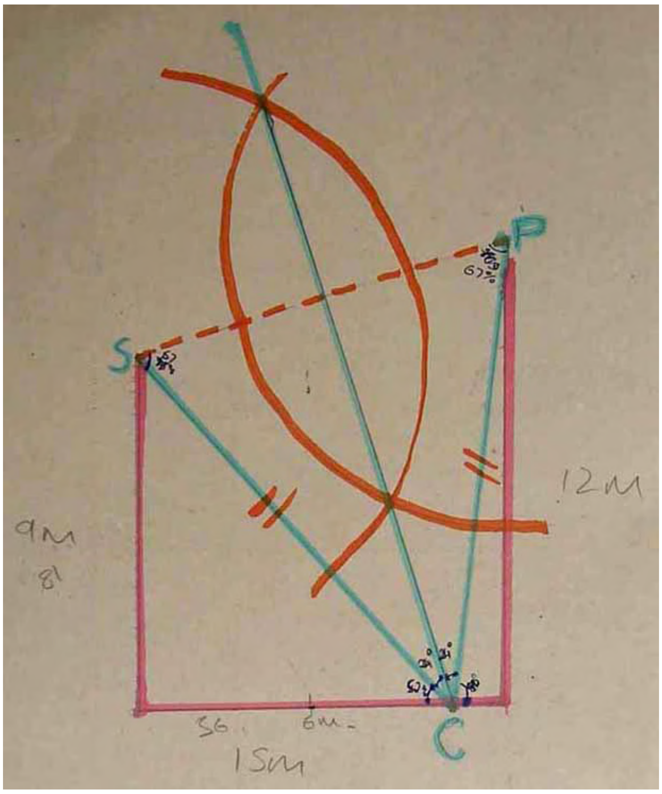
**** The task is attributed to Heron of Alexandria who lived round about 100 CE. Heron noticed that light travelling from one object to another via a mirror goes by the shortest route. This provides a useful context for thinking about the task.

A second common conjecture is to suppose the desired point is the midpoint of the projection of A and B on the given line *k* (below, left). Again, this is true when A and B are the same distance from line *k*, but imagine moving point A closer to the line (or even *onto* the line). It becomes obvious that the proposed position of P is not minimal (below, right).

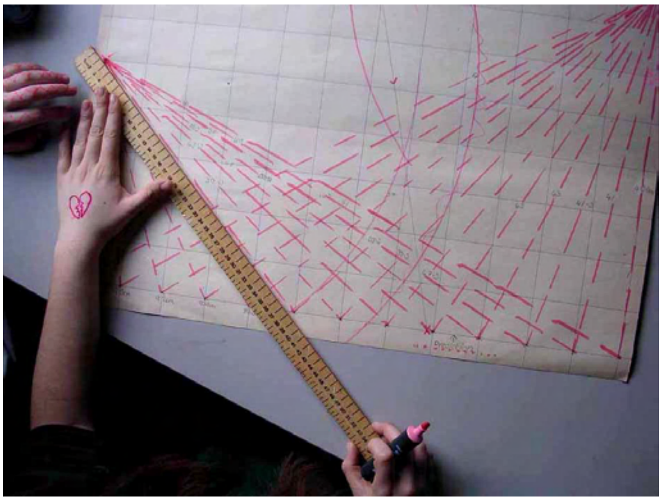


When we tried *Heron's Problem* in two project-school classes, students did occasionally make conjectures, as in the two cases below (from a Year 9 top set).* However, students were reluctant to interrogate their hunches, ie to conduct thought experiments. (This is not entirely surprising, as it is a difficult thing for the teacher to cultivate: it is often not practicable for the teacher to engage in such discussions with individual groups of students, and students may feel threatened if their ideas are put up for scrutiny by the whole class, unless an 'enquiry maths' culture has been developed.)

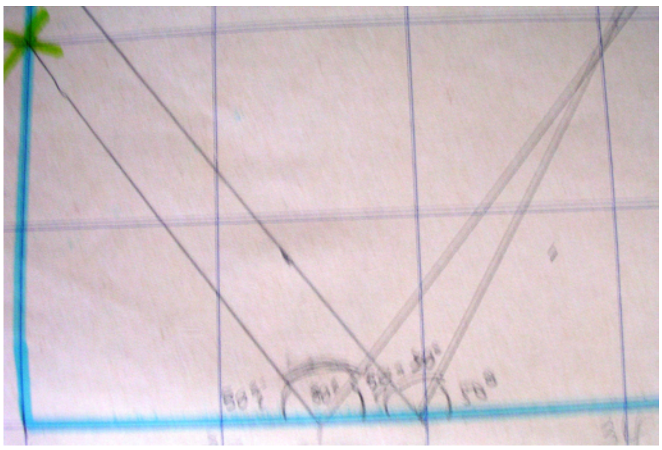




Generally, students preferred to seek the minimal point by making measurements, as in the case of a pair of students (from a Year 10 top set) whose work is shown below. As with the Year 8 student's work on the *Maxbox* task discussed earlier, these students were very systematic. However, they too seemed not to be evaluating their results as they went along. Initially they measured the lengths of the pair of lines that they drew, in order to find the total distance. But they soon adopted a 'production line' approach, whereby they drew the pairs of lines for *all* the grid points on the base line (line *k*) and only then measured their lengths to calculate the distances.

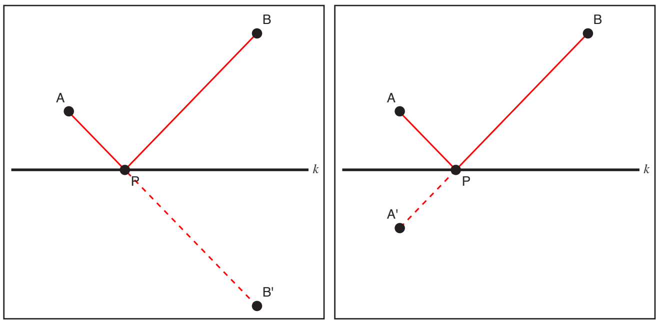


One pair of girls arrived at the conjecture that the shortest distance would occur when the angles that the lines AP and BP make with line *k* are equal. This turns out to be true, though they were not able to prove this or go on to find the desired point analytically. They called the 50˚ angles (below), angles of incidence and reflection - interestingly, they had recently discussed reflection in a physics lesson.



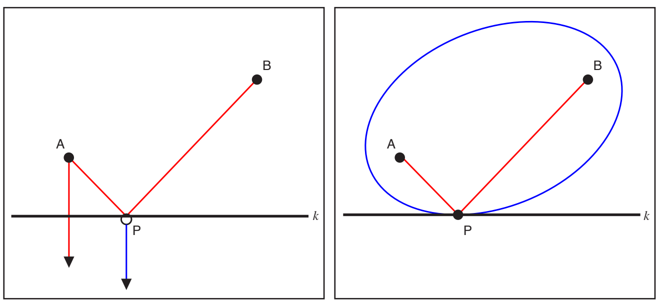
* The teacher had contextualised the task as one of finding the best place to put a computer (along a given wall) if one had to make frequent journeys from a scanner to the computer and then to a printer.

It turns out that a nice way to solve the task *is* by using the idea of reflection, and presumably this is what Heron had in mind. If one reflects point A or point B in line *k* and then draws a straight line from the image to the other point, then this line intersects line *k* at the minimal position of P (below left and below right). This makes use of a very familiar piece of mathematics and something which one might *expect* to be relevant (namely, that the shortest distance between two points is a straight line), but it also requires a creative leap (to identify the necessary points). It could be said the two girls came nearest to seeing this, thanks to a recent experience recalled from their physics lesson, but no one else came close, not even ‘retrospectively’, ie after finding the minimal point empirically.



It is difficult to know how students can be taught to make such leaps - but it is worth encouraging students to at least *look* for them (rather than devote all their attention to finding data), and also to consider what other mathematical knowledge might be relevant.

Interestingly, the task can also be solved using knowledge about weights and string* (below, left) or ellipses** (below, right).



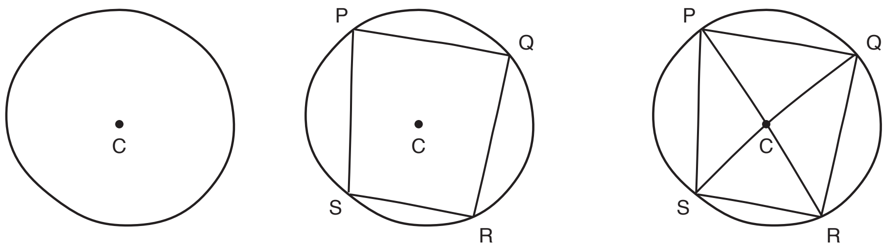
* A string is tied to a pin at B, passes through a frictionless ring with a weight attached, and then over a pin at A; the weight is lifted by pulling on the string, until the ring touches line *k*.

** The ellipse has foci A and B and is tangential to line *k*.

Darren sketches a circle. He calls the centre C.

He then draws a quadrilateral PQRS, whose corners lie on the circle.

He then draws the diagonals of the quadrilateral.



Darren says
“Whatever quadrilateral I draw with corners on a circle,
the diagonals will always cross at the centre of the circle”.

Is Darren right? . . . .

Explain your answer.

Question G1 (above) was used in the Year 8 and Year 10 Longitudinal Proof Project tests. The given statement can very simply be shown to be false, by using a thought experiment, namely by imagining moving one of the points P, Q, R or S slightly, to another position on the circle: if the diagonal from the moved point passed through the centre of the circle originally, it will no longer do so.

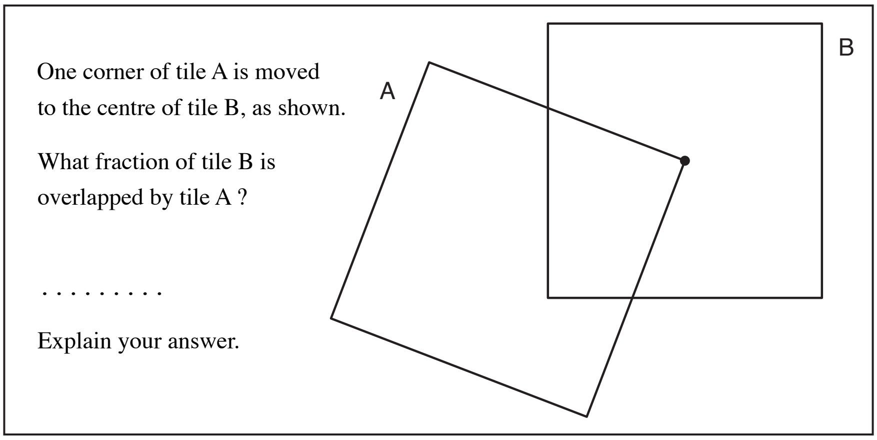
In the event, we found very little evidence of students using this strategy. Two fifths of our high attaining sample of 1512 students agreed with the statement in Year 8, with one quarter still doing so in Year 10. Students who answered correctly usually provided a drawing of a counter example. Often this would happen after a series of seemingly random trials, with students producing drawings of several quadrilateral to see what happens.

On the other hand, it is encouraging to note that a substantial minority of students (19 % of the sample in Year 8, 32 % in Year 10) gave answers of the type below, some of which might well have stemmed from thought experiments.

Because all the corners can touch above the centre of the circle

eg.

How does one encourage students to think in a more analytic/dynamic way? We have said a bit about this in chapter 2, in relation to *Euler’s relationship* (p2.6) and the *Angle sum of a triangle* theorem (p2.7); and also with regard to drawing and understanding diagrams in chapter 4 (p4.10). A way forward might be to invent tasks where a slight variation (such as moving a vertex under a given constraint) destroys a mooted relationship; and tasks where the relationship is preserved*. (for example, consider an equilateral triangle with two fixed vertices and one that can move (so that two of the sides can change in length): will the angle at the free vertex stay at 60˚ if the vertex moves parallel to the fixed side? If not, what might the path be?]

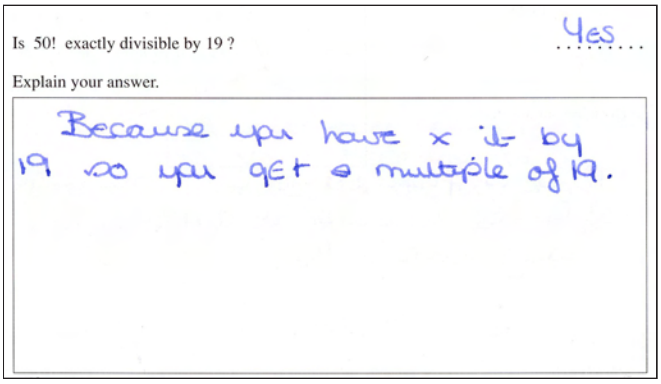


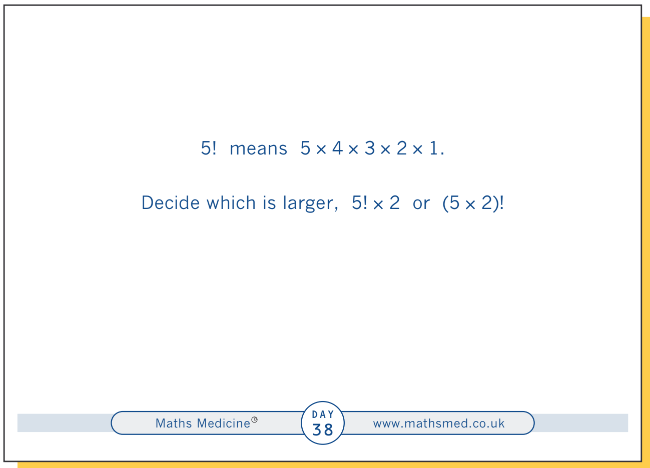
Item G2b (top right of page) was used in all the Longitudinal Proof Project tests. It shows two identical squares with the corner of one at the centre of the other, and asks students for the area of the overlap. Most of our students could determine the required fraction (a quarter) correctly. However, many had difficulty with their explanations, which ranged from the perceptual ("It looks like a quarter") to the analytical ("If you turn square A, then the triangular region that is uncovered has the same area as the triangular region that is newly covered").

The item provides a good starting point for conducting thought experiments. For example, it can be used to explore the givens in a situation, by examining what happens when the givens are changed: moving the pivot point, changing the size of the rotating square, changing the squares to, say, rectangles. How do such changes affect the overlap, and what (if anything) remains invariant? Thus one might ask, "For what pairs of shapes (one fixed and the other rotating about a vertex placed at the centre of the other), does the area of overlap stay the same?".

We also used a question (A4) concerning factorials. In the Year 9 version, after defining *factorial*, we asked students whether 5! is divisible by 3 and then whether 50! is divisible by 19. Most students (83% of the sample) answered the first part correctly, but nearly all of them did so by evaluating the factorial (5! = 120, 120 ÷ 3 = 4). This of course is not possible with 50!, which is why we chose it. We wondered whether students could forgo closure and analyse the structure of the situation. In the event, very few did so, with only about 9 % of our Year 9 sample giving a response like the one at the bottom, left of the page. Instead, students would try to evaluate 50! (eg it is 10 × 5! = 1200), or they would give spurious mathematical arguments like, '50! is not divisible by 19 because 19 is prime'.

Unfortunately we don't have much evidence, either from the current project or the previous one, that students readily conduct thought experiments in mathematics. However, we are not convinced that this is due to thought experiments being intrinsically difficult. Unless clear evidence emerges to that effect, it is surely worth giving students more experience of thought experiments. Thus, we should encourage them to make conjectures about mathematical situations and to think these through, rather than immediately perform calculations or in other ways generate data. As with the above question on factorials, one way of doing this is to use tasks where it is impossible (or difficult or unnecessary) to generate data, or where, at most, 'partial' calculations are needed. Examples of such tasks are shown below.





Comment: In the first expression 5! is multiplied by 2, in the second it is multiplied by 10 × 9 × 8 × 7 × 6.

* For example, consider an equilateral triangle with two fixed vertices and one that can move (so that two of the sides can change in length): will the angle at the free vertex stay at 60˚ if the vertex moves parallel to the fixed side? If not, what might the path be?

21 × 31 = 651.

Use this to help
you calculate

22 × 32.

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Comment: 22 × 31 is 31 more than 21 × 31 (which is 682); 22 × 32 is 22 more than 22 × 31 (which is 704).

The average age of the 11 players
in a hockey team is 9 years 2 months.

The average age of the players and
their manager is 10 years 2 months.

How old is the manager?

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Comment: The first statement is equivalent to there being 11 people all aged 9 years 2 months. The second statements is equivalent to having 11 people who are all one year older, and an extra person aged 10 years 2 months...

The average of 8, 10 and 21 is 13.

What is the average of 8, 10, 21 and 13 ?

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Comment: We can infer that 8 + 10 + 21 is three 13s; so 8 + 10 + 21 + 13 is four 13s.

Which is larger, 21 ÷ 5 or 21.1 ÷ 5.1 ?

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Comment: The same amount has been added to 21 and to 5, so the *proportional* increase is not as great for the numerator as for the denominator...

How much larger is 26 plus twice 29
than 29 plus twice 26 ?

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Comment: One approach would be to compare both expressions to “26 plus twice 26”.