7 Evaluating students' responses

In earlier chapters we have written about the benefits (for teachers and students) of sharing and evaluating students' responses. Commonly this takes place as and when responses arise during a lesson. However, such activity can also be planned in advance, by compiling banks of responses and then selecting particular types for students to evaluate. It is this approach that we consider in this chapter.

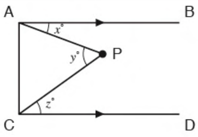
The selected responses need not have been made by the students engaged in the evaluation, indeed it can be beneficial, initially at least, to take responses made by other students (eg from a similar class, and carefully anonymised), so that the students undertaking the evaluation don't feel vulnerable or too inhibited.

The task below comes from a written test devised for the Longitudinal Proof Project. It was designed to investigate students' understanding of proof but it can equally well be used as a teaching activity, to acquaint students with different proof types and to help them identify their different characteristics. Several teachers on the current project did this, using this or similar tasks (eg the 'sum of two evens' task on page 2.2).

Unlike the items below, most of the items used in the Longitudinal Proof Project were open-response. For some of these we developed teaching activities based on examples of different response-types that arose in the research.

In the diagram, line AB is parallel to line CD, and AC is at right angles to both lines.

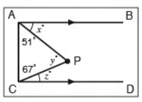
Points A, B, C and D are fixed.
Point P can move anywhere between AB and CD, but stays connected to A and C (the straight lines PA and PC can stretch or shrink).



Astrid, Burt, Cleo, Dilip and Emma are discussing whether this statement is true:

$x° + z°$ is equal to $y°$.

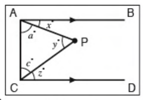
Astrid's answer



I could have a triangle APC with these angles. →
Then
$y = 180 - 51 - 67 = 62$,
$x = 90 - 51 = 39$, and $z = 90 - 67 = 23$.
But $62 = 39 + 23$, and as
$180 - 51 - 67 = (90 - 51) + (90 - 67)$,
I could have a triangle with other angles.
So $y = x + z$.

So Astrid says it's true

Burt's answer



The angle sum of triangle ACP is 180°, so $y + a + c = 180$.

Angles A and C are 90°, so I can write $90 - x$ for a, and $90 - z$ for c.

So $y + (90 - x) + (90 - z) = 180$,
so $y - x - z = 0$,
so $y = x + z$.

So Burt says it's true

Cleo's answer

I measured the angles in the original diagram.
I then moved P to another place and measured the angles again.

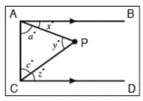
I made this table:

x	z	y
21	36	57
17	32	49

So both times I found that $x + z$ equals y.

So Cleo says it's true

Dilip's answer



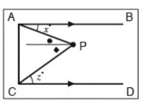
The angle sum of triangle ACP is 180°.

So I can write $a + c = 180 - y = 180 - (x + z)$.
Also $y = 180 - (a + c)$.

So $y = 180 - (180 - (x + z)) = x + z$.

So Dilip says it's true

Emma's answer



I drew a line through P parallel to lines AB and CD.

The new line cuts angle y into two parts.
The top part (●) is equal to x because the new line is parallel to AB. The bottom part (◆) is equal to z because the new line is parallel to CD.
So, altogether, y is equal to $x + z$.

So Emma says it's true

a) Whose answer do you like best? . . . .

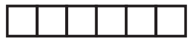
b) Whose answer is closest to what you would do? . . . .

c) Whose answer would get the best mark from your teacher? . . . .

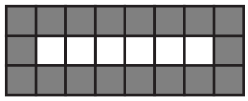
To get a sense of what students can get from such an activity, we focus in this chapter on a discussion I had with four high attaining Year 8 students on selected responses to Question A1, shown below. As mentioned in Chapter 2 (pages 2.3 an 2.4), A1 involved a familiar kind of pattern of white and grey tiles, and

A1 Lisa has some white square tiles and some grey square tiles.
They are all the same size.

She makes a row of white tiles.



She surrounds the white tiles by a single layer of grey tiles.



How many grey tiles does she need to surround a row of 60 white tiles?

Show how you obtained your answer.

was designed to see whether students could discern the pattern's structure and use this to make a 'far generalisation', ie to determine the number of grey tiles needed for a large number of white tiles, in this case 60. The numbers of tiles in the item were deliberately chosen to see whether students would resist operating solely on the basis of number patterns. Thus in the given array, there are 3 times as many grey tiles (18) as white tiles (6). Also, there are 10 times as many white tiles in the desired array (60) as in the original array (6). An argument based solely on these number patterns leads to the conclusion that 180 grey tiles are needed (rather than 126) for 60 white tiles. We found that a sizeable minority of students in the Longitudinal Proof Project gave this answer, either using the argument that there should be 10 times as many grey tiles in the required array as in the original array (ie 10 × 18 = 180), or (less frequently) there should be 3 times as many grey tiles as white tiles in the desired array (ie 3 × 60 = 180).

Response E below shows the 10×18=180 argument and is one of several responses that we printed on card,

How many grey tiles does she need to surround a row of 60 white tiles? . . 180 . . .

Show how you obtained your answer.

6 white tiles = 18 grey tiles
6×10 = 18×10
60 = 180

E

for use in our discussion about A1 with the four Year 8 students. In all, the discussion took about 20 minutes. We started by asking the four students to attempt the item themselves and then showed them card E with the ×10 number pattern spotting response. This is how the discussion proceeded:

DEK (interviewer): [reads out response] What's going on there... can you make sense of that... whether it's right or wrong is another matter... can you kind of read their mind...?

Group Yeh, I can see...

Ella ...well if you've got 6 white tiles... and 18 white (sic) tiles and 6 times 10 equals 60 so that means that you have to do 18 times 10 to get your answer... I can actually see me doing that...

DEK: Can you?

Alan I didn't actually look at how many white tiles there were here, I didn't notice how many there were...

DEK You were lucky, maybe...

Alan ...they've included the 3 at the end but they will always be 3, they won't get any bigger.

DEK They'll always be 3....?

Alan Those ones will always be 3 at the end... ...as long as it's just one row [of white tiles] there will always be 3 and it is only these ones [the horizontal rows of grey tiles] that change.

DEK Right. So what is this [18×10] answer doing to those 3 at the end?

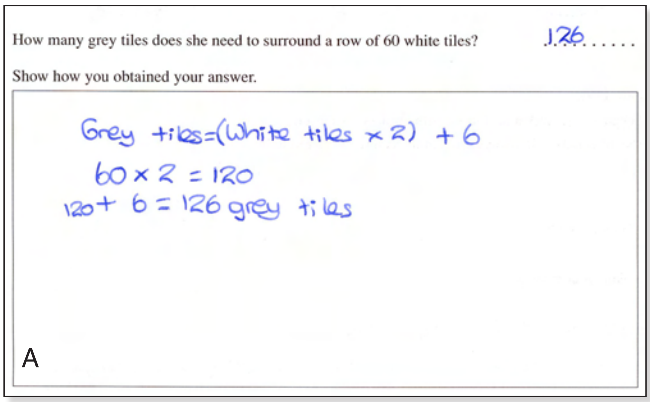
Alan multiplying them by 10... whatever it is... well it is including them... ... they should only multiply... what is it? -1, 2, 3, 4, 5, 6... they should only multiply 12 by 10... that amount is the only one that gets bigger... and that one.... these ones always stay as 3 on either end... so if you multiply them then it suggests it is getting bigger like that as well...

[Discussion continues]

This discussion gave the students the opportunity to make sense of and evaluate someone else's response. Rather than just dismiss the response as wrong, the students identified the method that had been used and tried to make sense of its inner logic. Moreover, Alan went on to consider *why* the method was wrong, by analysing the effect of multiplying all the original grey tiles by 10. (It is legitimate to multiply the row of 6 grey tiles above and below the original 6 white tiles by 10, but not the columns of 3 grey tiles at each end, as the resulting border of grey tiles would no longer fit snugly around the white tiles.)

Next, the four students were shown five more response-cards, A, C, D, F and I, below. All had the correct result, but the explanations/working differed in a variety of ways, for example in the use of words and diagrams, and in their explicitness. The discussion went as follows:

DEK Would anyone like to say anything about them - which one they like... or is similar to their own...?


How many grey tiles does she need to surround a row of 60 white tiles? ..126.....

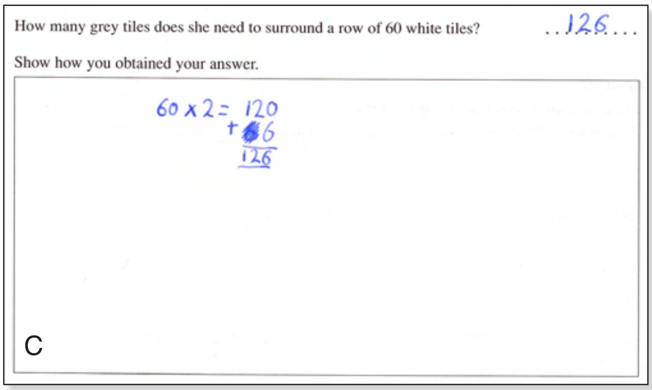
Show how you obtained your answer.

Grey tiles=(White tiles x 2) + 6

60 x 2 = 120

120+ 6 = 126 grey tiles

A


How many grey tiles does she need to surround a row of 60 white tiles? ..126...

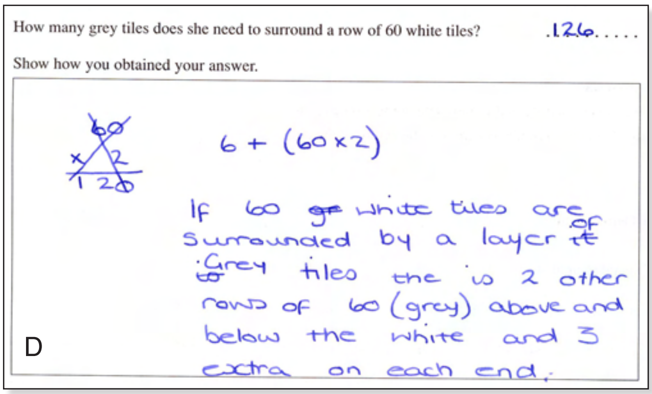
Show how you obtained your answer.

60 x 2 = 120

+ 6

126

C


How many grey tiles does she need to surround a row of 60 white tiles? .126.....

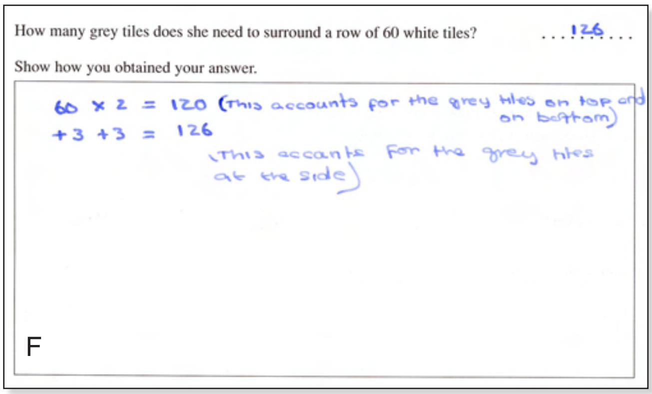
Show how you obtained your answer.

60 x 2 = 120

6 + (60 x 2)

If 60 white tiles are surrounded by a layer of Grey tiles the is 2 other rows of 60 (grey) above and below the white and 3 extra on each end.

D

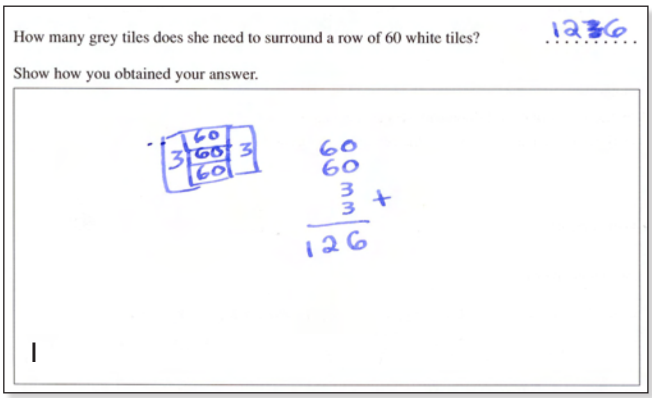

How many grey tiles does she need to surround a row of 60 white tiles? ...126...

Show how you obtained your answer.

60 x 2 = 120 (This accounts for the grey tiles on top and on bottom)

+3 +3 = 126 (This accounts for the grey tiles at the side)

F


How many grey tiles does she need to surround a row of 60 white tiles? .1236.

Show how you obtained your answer.

3 | 60 | 60 | 60 | 3

60 60 3 3 + 126

I

Hal I think that this one's the best [response I - which meets with general agreement], it's the best explanation, but I like that one best [C] 'cos its nice and simple.

Alan Also this one's interesting [A] because the person... they looked for a kind of formula... I did one, I did one [a formula] after I got the answer ... but they've done it by noticing the formula first and then working it out from that.

Ella I like that one [I], it's really simple and I think its a good idea to draw a diagram ['yes' from others in group].

DEK What about the fact that there are no words at all, so we've got a kind of...?

Joy That makes it clear.

Alan If you write it out with words, you have to kind of... you can write it kind of... it can be hard to explain how you did it... it can be hard to explain it, but if you do it with a nice clear diagram, it's really easy to see -

Hal - it's often difficult to relate words to maths...

[Discussion continues]

It can be seen that the students clearly preferred explanations that were simple and clear, and they were particular attracted by the visual explanation I, although none of them had used a diagram in their own attempt at A1 (as we shall see). Verbal explanations did not appeal to them, perhaps because they found them difficult to formulate and/or understand, as suggested by Hal's final comment, but perhaps also because such explanations tend not to be emphasised in the mathematics classroom. (At most, we tend to ask students to 'show working'.)

I continued the discussion by trying to probe whether the students could see any virtue in some of the more verbal explanations.

DEK OK, you might find it difficult ["to relate words to maths"], but if you think about someone communicating... say you had been writing this... for your teacher... or someone else who is not in the school so you wanted it nice and clear, or you were going to phone it - oh, that's a bit unfair.... because you then can't have a diagram... In terms of making it absolutely clear what is going on, do you still like the diagram best even though there is no kind of words to help us at all?

Alan - if they showed how the diagram relates to this [the working next to the diagram], if they showed that these two 60s [in the working] are the top two there [on the diagram] and the 3s are the two on the side

Joy - sometimes words muddle you up if you don't use the *exact* wording

Alan - also different people can interpret - if you explain it, different people can interpret it in different ways

DEK - is that not true of diagrams as well? Do you know that classic psychology diagram... symmetrical shape like that... some people see it as a vase and some people see it as two faces..?

Group - oh yes

Ella And there's one where you can see it as a really beautiful women or as a really ugly old woman -

DEK - that's right, yes

Ella [suggests a way of improving one of the responses (I?) but unclear]

DEK - oh I see...

Hal - this one [I] is best because you can relate it to any type of question

Alan [repeats his earlier suggestion of linking the diagram and the working in response I]

DEK Uh ha... Yes, I agree with you, it [I] is basically very nice and clear, no doubt about that... What about any of the others? This one's a formula [A], which is kind of elegant and nice mathematically, whether it explains it or not, I don't know...

Alan They've done it, like, 120+6 is 126 grey tiles, and they've put all the words in, like "grey tiles equals", not just .... they've put words in with the equation

Ella - I like that one [I], I think it's really clear... and I like these two [I and A], and I also like that one [C] -

DEK - So you like I and A and you also like C...

Ella - but those two [D and F] have got too many words

DEK - D and F have got too many words??

Alan Yes...

DEK If you had lots of time and ....

Alan - I think this one's [F] quite clear actually, 'cos it says um [reads] "60 times 2 is 120, this accounts for the grey tiles on the top and bottom" and then "plus 3 plus 3 is 126, this accounts for the grey tiles on the sides".

Ella Yes, I think it's clear but it's just a bit extra

Joy - a bit long... - especially if it was a test question or something as well

Alan - if they'd put a little diagram like that [I] in there, so you could see... 'cos also, when they say the ones along the top they might mean all those along the top including the ones at the sides, so if they put a diagram like that, and showed it... then I think it would be quite clear.

DEK OK, you're right about that, it depends what game you're playing. If you're doing a test you want to be quick but if you say have a period to write what you wanted to... So let's look at D and let's pretend they've got lots of time. [Reads D out loud.]

Hal I think that's a bit wordy really, it's a bit long

DEK - it's certainly wordy [laughter]... but is it a good explanation? - it may not be an efficient explanation but is it ...

Ella It's a good explanation but you can... it can just be written in a much simpler form just by doing that [I]

DEK - the diagram you think is just that much simpler

Ella - yes, it shows exactly that and it's much easier to take in.

[Pause in the discussion, while I consider showing some other response cards.]

It is clear that the students remained wedded to the clarity and directness of response I. However, the discussion did give them the opportunity to examine the other responses more carefully and to become more aware of some of their positive characteristics (as well as possible shortcomings, like the ambiguity of "on top"). Also, the discussion allowed the students to start thinking about the different purposes that a mathematical explanation might have and how this might relate to the audience to whom it is directed.

The students were then shown three more responses (J, K and L, below), all of which again had the correct result and again were quite diverse.

DEK ...OK, three quite different ones in a way... any thoughts?

Ella But I think that one, number J, is most like mine.

DEK Right. Is that the best or the worst!?

Ella I don't think it's the best... but it's just how I would do it.

Alan I think this one [K], although the diagram isn't labelled, I think it just helps you see that when they're saying these 60 - [edit] - even if it's not labelled, it helps you see that when they're talking about 'on each end' they're talking about those bits, and then... it just helps you see which bits they're talking about.

DEK OK... what about L? Any thoughts about that? Did you take in what that's about? - "for each white there is one above, one below"...

Ella I think with this person they just worked it out in their head and just wrote it down to explain -

DEK Yeh that's probably right, they did it afterwards to tell *us* how they did it...

Alan It's a bit like a formula, only worded, in a way... 'cos they say... for each line... for each white there are two grey tiles plus 6 on either end... on the ends... it's a bit like a formula.

DEK In what sense is it like a formula?

Alan Well it's a rule - it's a rule, saying for each white there will always be two greys... and then you just have to add on 6 for the ones at the ends. It could apply, that could apply for any number of white tiles... cos it's not like an equation, it's a rule that will always work for however many there are in the line -

DEK - right so, it's not just talking about the 60...

Alan - yeh it's... it uses 60 because that's how many there are but if you just change that and then did the maths for it it would work for anything.

DEK OK so if we call that sort of a verbal formula... and [respone] A was a formula, kind of, still had some words in it [Grey tiles = (White tiles × 2) + 6], and then you've got your own "$g = 2n + 6$" [*this was part of Alan's written response - see later*]. If we regard each of those as formulae: so yours, A and L. Any views about the three of them, are they all equally good?

[Longish pause... various murmurs...]

DEK Alan's typical algebra formula, this one [A] which has a few words in but I think we can still call it a formula, it's a rule for working it out for any number... and this [L] is a kind of verbal formula... where it is saying for each white you have to double, and then you have to add this 6 at the end...

Ella Well I like A the best. Sorry! 'Cos that's got too many words [L] and Alan's is kind of confusing, yeh, I don't understand algebra!

DEK Confusing?! Too few words?...

Ella Yes and so this one [A] is really simple, yeh, and that one [Alan's] I think is very mathematical...

Hal I like Alan's the best because I think it makes the most sense to me...

DEK Does it?? [some incredulity]

Hal - and it's the quickest and most efficient.

DEK OK, so it's a bit like the diagram in terms of quick and efficient...

Hal - and it's right, and it's the sort of thing I kind of thought of as well ...

DEK OK... that's fair enough... It's a nice way for you to summarise what is going on.

Ella For a mathematician, someone who is very good at maths it's a good one... for someone who likes a simple routine.

How many grey tiles does she need to surround a row of 60 white tiles? ...126...

Show how you obtained your answer.

60+60= 120 +3+3 =126

J

How many grey tiles does she need to surround a row of 60 white tiles? ..126...

Show how you obtained your answer.

60 white

60 grey on eachside
120
+ 3 greg at each end
6
126 grey tiles

K

How many grey tiles does she need to surround a row of 60 white tiles? .126...

Show how you obtained your answer.

for each white block there is ~~3~~ 1 above and below so I timesed 60 by 2 too find this number I then added 6 for the three blocks on each end.

L

DEK So it depends on who your audience is...

Ella - I think that one [A], to anybody who reads it, it would explain itself. To that one [Alan's] it would be very easy for people like mathematicians. And this one [L] is just long!

Alan If I was like talking to a class for example, I would use that one

DEK A?

Alan - yeh I would use A because it is kind of um...

DEK - what not L, you wouldn't use L?

Alan I would write that [A] and say that [L]

Hal I would summarise it in the end as that, though [Alan's].

Alan Yes... using A and L... I would say how I came to this [Alan's] formula and then finish with that

Ella It is the formula at the end that would get you good marks... [laughter].

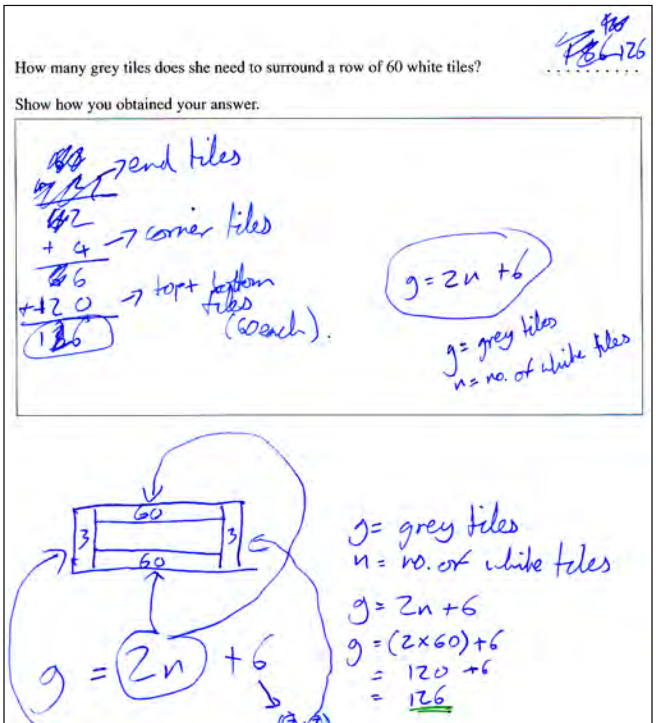
Alan is undoubtedly the strongest mathematician in this group, as is shown for example by his use of the formula "$g = 2n + 6$" in his written response for A1 and by the fact that he spotted the generality in response L ("For each white block there is 1 above and below"). Alan's use of language is interesting. In describing this generality, for example, he makes good use of phrases like "any number" and "however many". And he talks of the explanation being "a rule" and "a bit like a formula", both of which capture the sense of generality very nicely. However, it seems strange that he does not seem to have access to the term itself. Also, when he says it is "not like an equation" he is using 'equation' in an idiosyncratic way (he seems to be meaning a numerical statement or equality, such as $60 \times 2 + 6$, or $60 \times 2 + 6 = 126$).

Alan's insight with respect to response L may not have been fully shared by the other students, who accorded a high status to formulae involving algebraic symbols. As she had done earlier with response D, Ella dismissed L as containing "too many words"; however, she saw algebraic formulae as appealing to mathematicians (though not to her) and as attracting good marks. Hal aligned himself with an algebraic approach ("it's the quickest and most efficient" and "it's the sort of thing I kind of thought of as well"), even though his own response to A1 was purely numerical (see later).

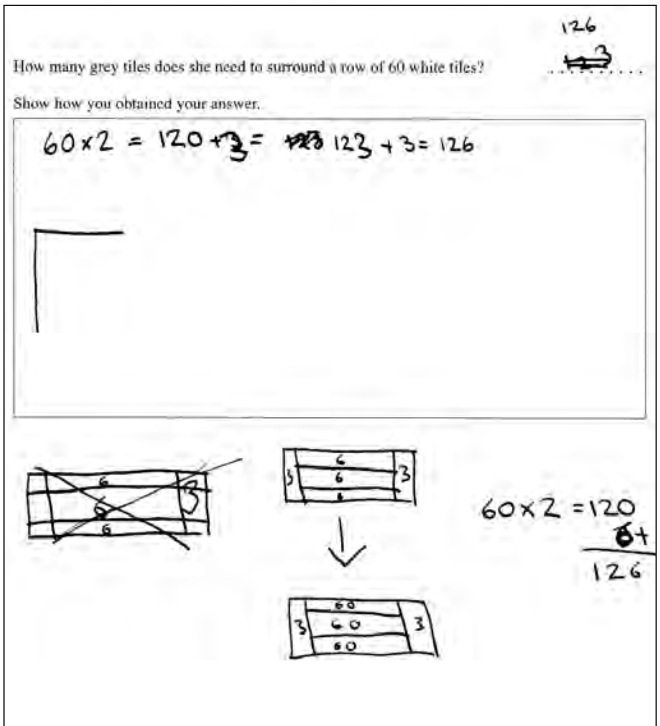
After the discussion, the students were asked to try A1 again, and to include in their revised explanation anything that appealed to them from the response cards and the discussion. Their original responses (in the answer box) and revised responses (written below the answer box) are shown below (for Alan, Ella, Joy and Hal respectively).

The most obvious difference between the pre- and post-discussion responses is that three of the students introduced a diagram into their explanations, no doubt influenced by response card I which they had been enthusiastic about during the discussion. The fourth student, Hal, used an algebraic formula in his second attempt, a form of explanation that he had strongly endorsed earlier. In line with the group's views about brevity and clarity, most of the explanations were still quite short, and contained few if any words.

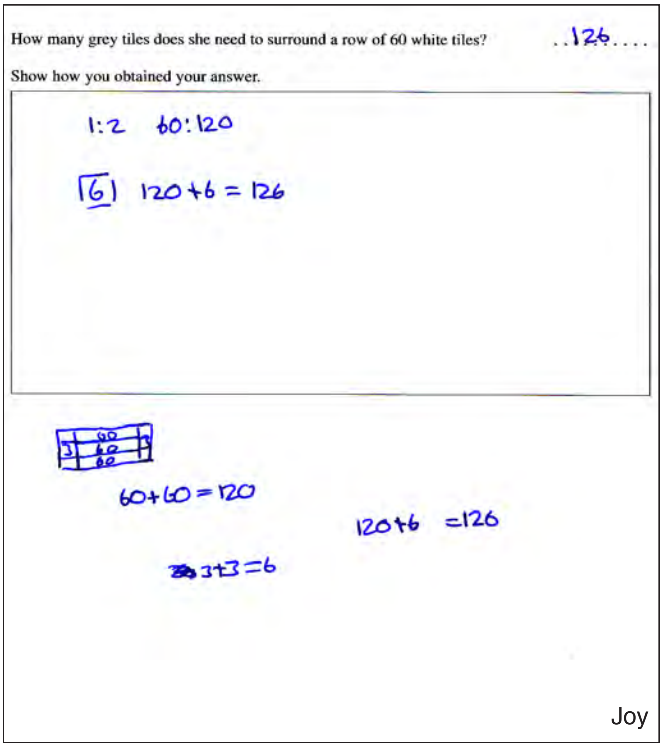
The discussion gave students the opportunity to think critically and insightfully about mathematical explanations. Using other students' responses made the situation non-threatening and made it possible deliberately to choose explanations that I thought might broaden their experience and provoke discussion

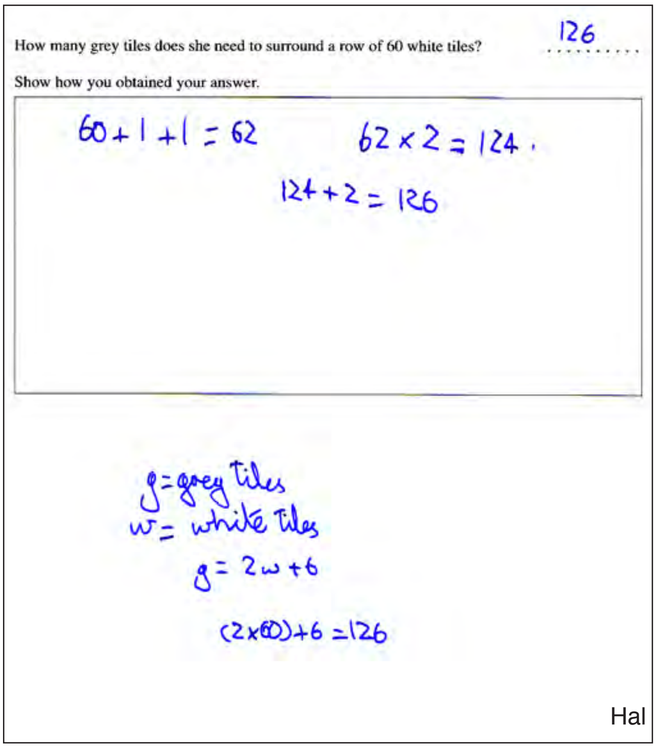


Alan



Ella





and reflection. The students had strong views about what they liked and disliked but they were receptive to at least some features of explanations that they had not used themselves. Also, it was possible to begin to shift the students' perspective from one of ranking explanations in terms of these likes and dislikes to a more dispassionate consideration of the quality of different explanations. They also began to think about the different functions that an explanation can have and how this might relate to its intended audience.