2 Choosing tasks and approaches to proof

Teachers on the project more often used arithmetic/algebra tasks than geometry tasks. In part this perhaps reflects the place of these areas within the National Curriculum, as well as the teachers' own knowledge (none of them were of an age where, for example, they would have encountered a systematic treatment of Euclidean proofs when they were at school). However, it may also be that, at school level at least, it is easier to engage in numeric rather than geometric arguments. Thus one can get quite a long way with a basic understanding of notions like odd and even and multiple; of course arguments in this area are likely also to assume things like the commutative and associative law, but these can often be left implicit.

In geometry, on the other hand, students can rapidly be faced with a complex web of relationships, where it is difficult to sort out the givens and to determine which theorems can be assumed, and which can easily lead to circular arguments. There are also difficulties associated with diagrams: what is the status of a diagram, is it a sketch or is it an accurate representation, does it represent a specific example or is it generic, can we take what appears to be true in the diagram to be true and use this in our proof? Students had considerable difficulties in producing diagrams that a. satisfied the givens (eg where a given isosceles triangle actually looked more or less isosceles) but that b. were also generic, ie did not include fortuitous relationships (eg angles or sides looking to be congruent in the specific diagram when this is not necessarily the case in general), and c. they had difficulty analysing the diagram into properties that we know to be true because they are given (whether they look it or not) and other properties whose truth may only be determined by deduction, whether or not they appear to be true from the diagram.

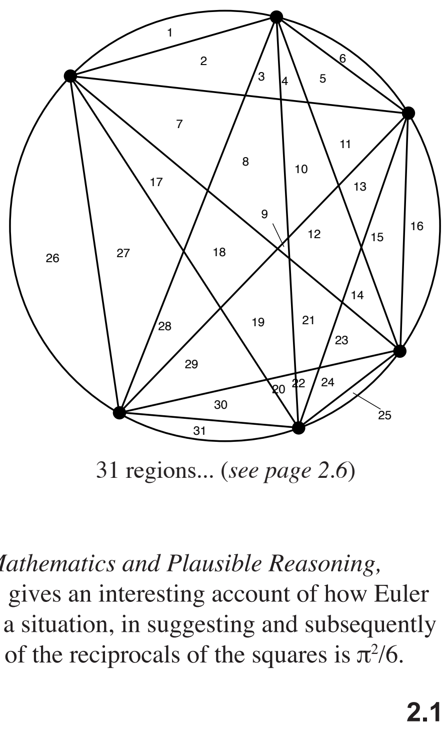
Also, simple situations in geometry can appear 'too simple', eg being asked to prove that the base angles of an isosceles triangle are equal. This relationship is so immediate that it is difficult to find the constituent parts needed to build a logical argument. It just is. Consider the proof where a construction line is drawn from the third angle to the midpoint of the opposite side, creating two triangles which are congruent because of SSS. Here, SSS seems more complex than the relationship we are trying to prove. Or take a proof where one of the equal sides is 'folded' onto the other. 'Obviously' the two base angles exactly coincide! How can it be otherwise?

By contrast, a simple numerical relationship like 'The sum of two odd numbers is always even', though we all know it to be true, is not as 'obvious'; here we need to reveal the structure of even and odd numbers (eg even numbers can be split into two identical whole numbers, while odd numbers have an extra 1); this may or may not be trivial, but the resulting argument, rather than being merely 'obvious' can be quite illuminating. (On the other hand, the relationship 'The sum of two even numbers is always even' *is* perhaps too obvious, putting it on a par with the isosceles triangle relationship.)

Looking for structure

The central idea of all our work is that to produce a mathematical explanation or proof we need to look for the structure inherent in a situation. It is not 'enough' (ie efficient, sufficient, satisfying, illuminating) just to find empirical relationships in numerical data, unless numerical data is all we have*. Students' success in doing this varied enormously, between and sometimes within classes. Where students had difficulties this may in part have been due to the cognitive demands involved, ie structure may sometimes be difficult to see, and the relationships difficult to disentangle. However, students' difficulties may often have arisen for a simpler reason, namely that they were not familiar with this way of seeing mathematics, in terms of structure. Thus students difficulties might, at least in part, be due to a lack of initiation. This, we think, is an important observation and one that teachers can do something about with relative ease. How far it will take their students remains to be seen, but it would seem to be an obvious and extremely worthwhile first step for teachers to take.

Of course, there are many ways of setting about looking for structure, and so it is important that students and teachers learn to recognise them. However, equally important, and another key idea that ran through our work, is that teachers and students need to become more aware that different tasks lend themselves to different approaches.



31 regions... (*see page 2.6*)

* Polya (1954), in *Mathematics and Plausible Reasoning, Volume 1*, chapter 2, gives an interesting account of how Euler coped with just such a situation, in suggesting and subsequently proving that the sum of the reciprocals of the squares is $\pi^2/6$.

Proof prototypes and item prototypes

Question A3, shown below, is closely based on an item developed by Healy and Hoyles. This version comes from the Year 10 proof test that was used in the Longitudinal Proof Project. The question nicely illustrates the different ways that students might 'prove' a statement in number/algebra. Proofs A and C are both desirable prototypes, in that they involve a (correct) argument based on the (general) structure of even numbers, expressed in algebraic form in A and in narrative form in C. Proof F uses a visual form, but is also concerned with the structure of even numbers, in that even numbers are shown as consisting of two equal parts (although the degree to which the structure is seen as *general* is uncertain, as it depends on whether the observer sees the arrays of dots as representing 'any' even numbers, or the specific numbers 10, 8 and 18). Proof B is empirical; it presents six examples each of which confirms the statement; there is a hint of generality about the examples in that they are presented in a fairly systematic way, suggesting we could carry on producing such examples, all of which would be likely to confirm the statement. However, we cannot be quite sure that the statement is really true (or 'always true') and the examples provide no insight into why the statement is true. In other words, B neither fully convinces nor explains. Proof D has both empirical and general features. It says something about all even numbers, namely that (when written in base 10) they always end in 0, 2, 4, 6 or 8. But this property, and the similar property about the sum of even numbers, says nothing about the structure of even numbers, it is simply an empirical generalisation derived from observing a feature that is fairly immediately apparent. Argument D thus provides a proof that convincingly shows that the statement is true, but does not effectively explain why. Argument E has the appearance of generality because of its algebraic form; however there is no coherence about the set of statements and the last one is flawed, in other words E is a nonsense argument.

A3 Aysha, Brian, Coby, Deon, Eric and Fiona were trying to prove whether the following statement is true or false:

When you add any 2 even numbers, your answer is always even.

Aysha's answer

a is any whole number.
b is any whole number.
$2a$ and $2b$ are any two even numbers.
$2a + 2b = 2(a + b)$.

So Aysha says it's true

Brian's answer

$2 + 2 = 4$ $\quad$ $4 + 2 = 6$
$2 + 4 = 6$ $\quad$ $4 + 4 = 8$
$2 + 6 = 8$ $\quad$ $4 + 6 = 10$

So Brian says it's true

Coby's answer

Even numbers are numbers that can be divided by 2. When you add numbers with a common factor, 2 in this case, the answer will have the same common factor.

So Coby says it's true

Deon's answer

Even numbers end in 0, 2, 4, 6 or 8. When you add any two of these the answer will still end in 0, 2, 4, 6 or 8.

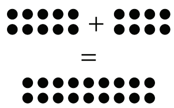
So Deon says it's true

Eric's answer

Let x = any whole number, y = any whole number.
$x + y = z$
$z - x = y$
$z - y = x$
$z + z - (x + y) = x + y = 2z$

So Eric says it's true

Fiona's answer



So Fiona says it's true

a) Whose answer do you like best? ....

b) Whose answer is closest to what you would do? ....

c) Whose answer would get the best mark from your teacher? ....

Question A3 was devised as a research question; however, it also makes an interesting classroom activity that allows students to interpret and evaluate the qualities of different kinds of arguments. In the Longitudinal Proof Project Year 10 proof test, A3 was followed by and served as a prompt to a question involving a similar statement which students were asked to prove themselves: "When you add any two odd numbers, your answer is always even" (see chapter 1) . The research confirmed what was becoming increasingly clear from other studies, that there was a strong tendency for school students to produce empirical 'proofs'. Thus, for example, 41 % of our sample of 1512 students chose the simple empirical 'proof' B as closest to their *own approach*. However, the picture is more subtle than that. When, for example, students were asked to select the proof they *liked best* in A3, many chose a different proof from the one that mirrored their *own approach*. The frequency for option B was now only 17 %, with option D (the general, but non-structural proof) now the most popular (it was chosen by 35 % of the sample). Moreover, 40 % of students chose one of the structural proofs (mostly A or C but occasionally F) as the one they *liked best*, compared to 25 % when asked for *own approach*. This suggests that though they might not have been sufficiently familiar with structural proofs to be able to construct them themselves, they could appreciate their general and/or structural qualities. (Of course, a sizeable proportion chose option E, especially for *best mark*, which also indicates that students can be mislead by the surface qualities of a proof.) In turn this suggest that helping students become more familiar with structural proofs and encouraging them to produce such proofs themselves could well bear fruit with many students. Indeed, this formed a major plank of our activities, as will become clear in later chapters.

However, the purpose of this chapter is rather different. Just as there are different kinds of ‘proofs’ or proof-prototypes, as Question A3 illustrates, so there are proof items which lend themselves to different initial approaches*. Thus in this chapter we want to highlight different *proof-item prototypes*. This is especially pertinent in the context of number/algebra where the tendency has developed (probably ever since ‘investigations’ became part of the examination system in the form of GCE/GCSE coursework) to tackle proof items in a standard way, namely

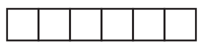
- work systematically
- generate data
- make a table
- look for numerical patterns
- describe the pattern (express in general form)
- explain the pattern.

We want to argue that though this is sometimes an effective way to tackle a problem, often it is not and instead it all to often becomes displacement activity resulting from a conspiracy (or didactic contract) between teachers and students (whereby students agree to produce a lot of ‘work’ and the teacher excuses them from having to think mathematically about what they are doing).

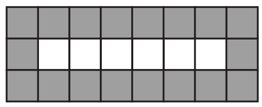
Consider item A1 (right) from our Year 8 and Year 10 proof tests, which involves a familiar tile-pattern context. In this item, students are asked about the configuration containing 60 white tiles, and so are immediately required to make a ‘far generalisation’. The item beneath it (concerning matchsticks) shows a more typical version of such a task (taken from Key Maths book 9.3). In this second task, students are presented with several, ordered configurations, starting with the smallest possible configuration. They are asked to draw the next two configurations and to put the resulting values (pattern number and number of matchsticks) into a table, and then (not shown) to find a rule for calculating the number of matchsticks. Such preliminary steps before making the generalisation are likely to be quite unnecessary for many students, given that one can see the relationship (between the numbers of white and grey tiles in our task, or between the number of hexagons and matchsticks in the case of the Key Maths item) from the geometrical arrangement of the tiles or matchsticks. We would further argue that the abstraction involved in going from the tile or matchstick pattern to putting numbers in a table may actually divert students from looking for the structure inherent in the pattern. Of course, it is possible that for some students and for some patterns, it helps to consider several configurations of the given pattern and perhaps even to draw some of the configurations oneself. Perhaps, if students want to do this, they should be allowed to do so (if only to let them discover that drawing a pattern can be fraught with difficulties and may sometimes be counter-productive!). However, it should be regarded as a fall-back strategy when one is well and truly stuck, rather than as the default first step.

A1 Lisa has some white square tiles and some grey square tiles. They are all the same size.

She makes a row of white tiles.



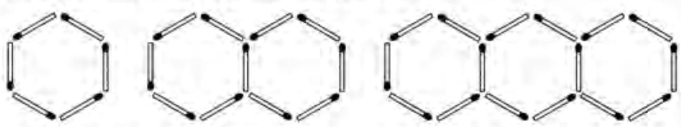
She surrounds the white tiles by a single layer of grey tiles.



How many grey tiles does she need to surround a row of 60 white tiles?

Show how you obtained your answer.

Here are some of Jane’s matchstick patterns.



a Sketch the next two patterns.
b Copy this table and fill it in.

Number of pattern	1	2	3	4	5
Number of matchsticks	6	11			

In the Longitudinal Proof Project research with high attaining secondary school students, 47% of our sample (N=1512) successfully made the far generalisation in question A1 in Year 8 (ie gave the answer 126 together with a satisfactory explanation or working). (These same students met the question again when they were in Year 10: the success rate increased to 70%, and there was also a marked increase in the quality of students’ explanations, with more references to variables and more use of algebraic notation.) Though these students were drawn only from the top one or two mathematics sets in their school, the success rates are high enough to suggest that it would be worth encouraging a far broader range of students to embark, from the start, on far generalisations for geometric patterns such as those in A1. In the event, some project teachers did precisely this with considerable success, as we discuss later.

* We are using the term ‘proof’ here in the manner of Harel and Sowder, ie not in a precisely defined mathematical sense but rather to denote any argument that the user might describe as a proof.

Our results on A1 are interesting for another reason. Even though students were shown only one configuration and asked to make a far generalisation straight away, we found that a sizeable minority of students seemed to ignore the geometric properties of the tile-pattern and focussed instead just on the numbers involved in the given and desired configuration (the given configuration has 6 white tiles and 18 grey tiles, and students are asked for the number of grey tiles in the configuration containing 60 white tiles). Thus 35% of the Year 8 students, falling to 21% in Year 10, gave the answer 180 to question A1, based on one of two number-pattern spotting strategies:

> either, 10×6=60, 10×18=180 (ie, there are 10 times as many white tiles in the new configuration as in the given configuration, so there will be 10 times as many grey tiles in the new configuration as in the given configuration)
>
> or, 6×3=18, 60×3=180 (ie, there are 3 times as many grey tiles as white tiles in the given configuration, so there will be 3 times as many grey tiles as white tiles in the new configuration).

We deliberately designed question A1 to contain numbers that would entice students to use these number-pattern spotting strategies. However, we still think these percentages are very high, given the nature of our sample. It seems likely that many of our students were susceptible to these number-pattern spotting approaches at least in part because of the way they had been taught. This reiforces our argument that students should be made more aware of alternative strategies, in particular that of focussing directly on the structure inherent in the context.

By way of contrast, we now consider a task where the systematic generation of data *is* an effective way of starting:

Which products of 3 consecutive numbers are multiples of 24 ? Explain why.

This task can of course be tackled by thinking about structure from the outset* (though we have never observed anyone doing so!). Here, it seems, it is hard to resist considering at least one or two cases, eg 1×2×3=6 and 2×3×4=24. Continuing in this way, one soon discovers that products that start with an even number are always multiples of 24. It is then tempting to think that this has answered the first part of the task, ie that we have identified all cases giving a multiple of 24. In fact, some products which start with an odd number also give a multiple of 24, for example 7×8×9=504. These cases can easily remain undiscovered unless one generates a fairly large and systematic set of examples. A particularly effective way of doing this is with a software package like Excel, where it is relatively simple to construct formulae to evaluate products and to test whether they are multiples of (or divisible by) 24 and where these can rapidly be applied to many cases using the copy down facility. Such a table of data is shown below.

A	B	C	P=A×B×C	P/24
1	2	3	6	0.25
2	3	4	24	1
3	4	5	60	2.5
4	5	6	120	5
5	6	7	210	8.75
6	7	8	336	14
7	8	9	504	21
8	9	10	720	30
9	10	11	990	41.25
10	11	12	1320	55
11	12	13	1716	71.5
12	13	14	2184	91
13	14	15	2730	113.75
14	15	16	3360	140
15	16	17	4080	170
16	17	18	4896	204
17	18	19	5814	242.25
18	19	20	6840	285
19	20	21	7980	332.5
20	21	22	9240	385
21	22	23	10626	442.75
22	23	24	12144	506
23	24	25	13800	575
24	25	26	15600	650
25	26	27	17550	731.25
26	27	28	19656	819
27	28	29	21924	913.5
28	29	30	24360	1015
29	30	31	26970	1123.75
30	31	32	29760	1240
31	32	33	32736	1364
32	33	34	35904	1496

From the table we can readily induce that the product is a multiple of 24 when the first number is even, or when the middle number is a multiple of 8. (There are obviously other equivalent formulations, eg "the middle number is odd or a multiple of 8", or "the first number is even or 1 less than a multiple of 8".) Thus we can now confidently assume that we have all the cases, ie a complete answer to the first part of the task. However, it still remains to explain why this result is true, which means looking for structure in the sets of multiples that fit the rule. This may still be quite a challenging task, depending on how familiar students are with the properties of consecutive numbers (eg, given a multiple of 3, every 3rd subsequent consecutive number is a multiple of 3, etc).

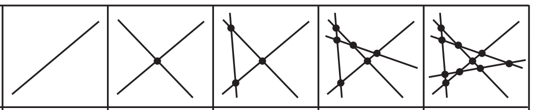
We next consider a task which is commonly tackled by building up a sequential set of cases, but which can be solved more directly by considering a generic case:

Some (infinite) straight lines are drawn so that every line intersects every other line. How many points of intersection are there for *n* lines? Explain why.

This can be solved inductively by considering the number of intersections for 1, 2, 3, 4, etc lines and finding a rule that fits the resulting numbers. Thus students might start by drawing diagrams and putting

* A structural argument might go like this: If I have three consecutive numbers, one of them must be a multiple of 3. So if one of the numbers is a multiple of 8 (eg, 8×9×10, 7×8×9, 6×7×8, etc) I will always get a multiple of 24. I will also get a multiple of 8 (and hence of 24) if the first number is even as this means that the last number will also be even and that one of these must be a multiple of 4 (with the other a multiple of 2).

the results in a table (see below). However, this approach has several pitfalls. First, the diagrams can be difficult to draw - how does one know that every line has been drawn to cross every other line? How



Number of lines	1	2	3	4	5
Number of intersections	0	1	3	6	10

does one know that every point of intersection has been counted? Then, unless one knows some technique like the method of differences, how does one find a rule that fits the numbers? Also the task of drawing, counting, and recording can easily turn into a displacement activity, ie it becomes an excuse for not thinking about structure and not striving for insight.

By generating successive terms or configurations, students may end up adopting an iterative approach, whereby they compare one term with the next. This can lead to insight, and to an efficient rule for the nth term, but it often does not. In the present task, the number of intersections increases by 1, 2, 3, 4..., as can be seen from the table above, ie the 2nd line produces 1 extra point of intersection, the 3rd line 2 extra points, and so on. This might lead to the insight that the 10th line, say, produces 9 extra points and the nth line $(n – 1)$ extra points, but it is also likely to result in a rule for the total number of points that is expressed 'iteratively' rather than in a 'closed' form. Thus it is likely to lead to the expression $1 + 2 + 3 + ... + (n – 1)$ for the total number of points of intersection (where the number of terms in the expression depends on the value of n), rather than the closed expression $n(n – 1) \div 2$. (Of course, we can derive the second expression from the first by using our knowledge of arithmetic progressions or by, say, writing the first expression again in reverse order and summing aligned terms...)

An alternative approach is to *look at the problem* (as Polya puts it), ie to look for structure rather than data, from the very start. And for this particular task it also helps to look at the set of lines in a generic (or static) rather than an iterative (or dynamic) way*. Given that every line cuts every other line, then if there are 10 lines, say, every line cuts 9 other lines; or if there are n lines, every line cuts $n – 1$ other lines. That is the key. And so there are 10×9 points of intersection altogether for a configuration of 10 lines - except that we have to divide by 2 because every point is generated by two lines and we would otherwise be counting points twice. And so for n lines there are or $n(n – 1) \div 2$ points.

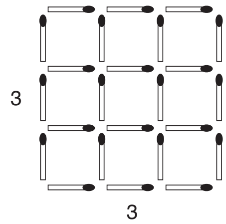
It should be emphasised that we are not trying to say that we, or our students, must solve particular problems in particular ways. Rather, we want to develop an awareness, for ourselves and for our students, that there are different ways of solving problems and that it may turn out (even if we can't always tell in advance) that some ways are particularly well suited to solving some problems. 'Making a table' has become a default strategy in many classrooms, from notions that are perfectly sound but which don't always offer the best course of action: for example, the notion that one should 'explore' or 'get a feel for' a situation or problem and that one should be systematic and 'look for patterns'.

We consider now another example of a task which is often solved inductively (ie by generating data and looking for patterns) but which again, like the intersecting lines task, can effectively be solved generically, by *looking at the problem*. The task concerns a square grid of matchsticks:

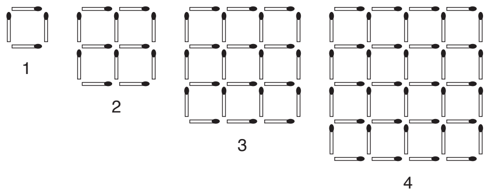
This is a 3 by 3 square of matchsticks.

How many matchsticks are needed for

- a 20 by 20 square
- an n by n square?

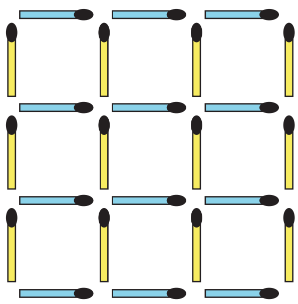


A common way to solve this task is to draw several matchstick arrays and to construct a table, with the hope of finding a rule for the numbers in the table:



width	total
1	4
2	12
3	24
4	40

However, as with the previous task, one can also approach it by simply *looking at the problem*, ie by taking, say, the 3 by 3 array and treating it generically. Thus for example, one can construe the 3 by 3 array as 4 rows of 3 matches and 4 columns of 3 matches, or, for a 20 by 20 array, 21 rows of 20 matches and 21 columns of 20 matches, or, for an n by n array, as $(n + 1) \times n \times 2$ or $2n(n + 1)$ matches.

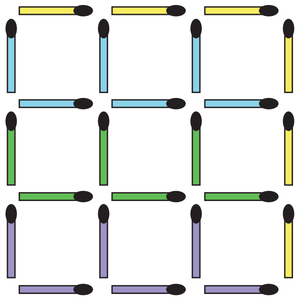


4 rows of 3 matches and ...

What is also nice about this task is that one can construe an array in different but equivalent ways. For example, the 3 by 3 array can be thought of as 3 rows of 3 L-shapes, where each L is made of two matchsticks, plus a row and column of 3 matchsticks along the top and the right hand side, which makes a

* Though there are tasks, such as proving Euler's rule about the numbers of faces, edges and vertices of polyhedra, where an iterative approach helps, as we discuss later.

total of 3 × 3 × 2 + 3 + 3, or, in the general case, $n \times n \times 2 + n + n$ matchsticks. The challenge now is to show that this is equivalent to the earlier expression, $2n(n + 1)$.

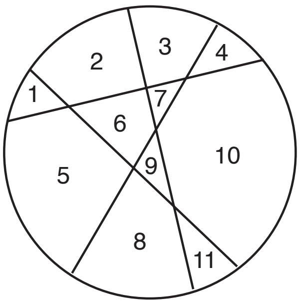


3 rows of 3 L-shapes and...

We now consider a task which at first sight is similar to the points-of-intersection task discussed above:

> **If *n* straight lines all intersect each other (inside a circle), how many regions do they make (inside the circle)? Explain why.**

The diagram (below) shows the situation where 4 lines have been drawn, which, as can be seen, produces 11 regions. (The circle is not really needed, but it means that all the regions are finite and 'closed'.)



4 lines, 11 regions

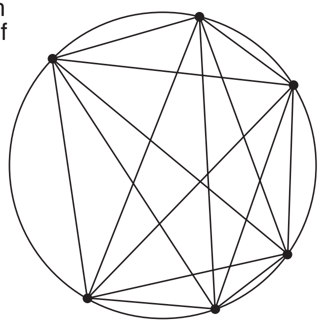
This task is less 'transparent' than the points-of-intersection task. There, if one had drawn 4 lines one could, with relative ease, see that each line intersects the 3 other lines, making 4×3÷2 points of intersection altogether. But how many *regions* does each line produce? The answer seems far from obvious. What is perhaps easier to see is how many *extra* regions are produced when a new line is added. For example, if a 5th line were added to the above diagram it would create 5 extra regions, because as it meets its first line it creates an extra region, as it does for each of the 4 lines that it meets, with the creation of 1 further region as it meets the circle again. This 'incremental' argument allows us to produce a rule expressed iteratively: when there are 0 lines there is 1 region; drawing the 1st line produces 1 extra region; the 2nd line produces 2 extra regions; and so on. So n lines produce 1 + 1 + 2 + 3 + 4 + ... + n regions. Now, with sufficient experience or insight, this can be turned into a closed expression, namely $1 + n(n + 1)÷2$. It will be recalled, a similar incremental approach is possible for the points-of-intersection task, where the nth line produces $(n – 1)$ extra points, giving 0 + 1 + 2 + ... + $(n – 1)$ points in all, which can be written as $n(n – 1)÷2$. However, as we argued earlier, this is not as intuitable or direct as the generic argument that each of the n lines intersects in $(n – 1)$ points, which makes $n(n – 1)÷2$ points in all.

Thus with the regions task, it seems that we cannot (at least, not easily) use a generic approach and so have to resort to an incremental approach instead. Alternatively, we could adopt a purely empirical approach, by considering lots of cases and searching for a rule that fits the resulting values. However, as we will see from the task we consider next, we can then never be sure that our rule will fit *all* possible values. This next task, sometimes called *Pancakes*, is even more intractable than the regions task:

> ***n* points are drawn on a circle. A line is drawn from each point to every other point. What is the maximum number of regions that can be formed inside the circle? Explain why.**

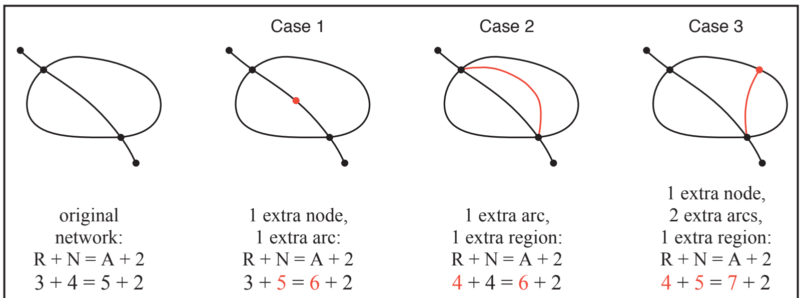
This task is much more challenging and a solution will not be given here*. If one considers the first few cases, the growth in the number of regions seems to fit a simple pattern - the number doubles each time (as shown in the table below). However, what is interesting from our point of view is that this pattern breaks down.

Number of dots	Maximum number of regions
1	1
2	2
3	4
4	8
5	16
6	?



If one carefully counts the number of regions in the 6-dot diagram shown above (which is not a trivial task), it turns out there are 31 regions, not 32 (see also the diagram on page 2.1). Thus *Pancakes* provides a useful (and dramatic) reminder that we need to take care when making generalisations on the basis of empirical evidence, without the support of a structural explanation.

Earlier (footnote 4) we mentioned Euler's Relationship which states that the number of faces (F), vertices (V) and Edges (E) of a simple polyhedron is given by the formula F + V = E + 2. The equivalent relationship for the number of regions (R), nodes (N) and arcs (A) of a simple network is R + N = A + 2.



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* A nice solution is given by S Hart in *Symmetry Plus*, 32, Spring 2007. See also Beevers (1994) Patterns which aren't, *Mathematics in School*, 23, 5 and Anderson (1995) Patterns which aren't are!, *Mathematics in School*, 24, 2.

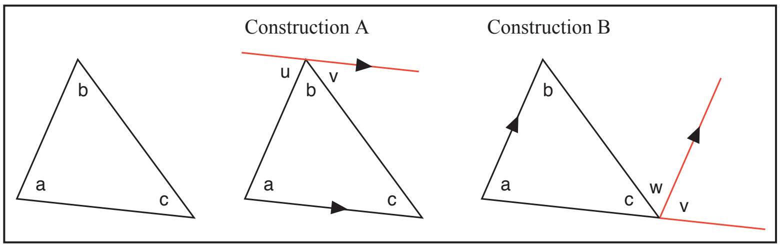
The relationship holds for the first network shown in the above diagram (if we include the 'outside' region). But how do we prove it is (always) true? Just as with the lines-and-regions task discussed earlier (but in contrast to the points-of-intersection task) it is difficult to see why the relationship holds simply by 'inspecting' a given (or generic) network. However, we can begin to make sense of the relationship by changing a network in incremental ways. Thus consider the changes made to our original network in the three cases shown above.

In each case, the changes are balanced: whatever change happens to the value of the left hand side of the formula, the same happens to the right. So if the original values of R, N and A satisfy Euler's Relationship, the new values do too. We have here the beginnings of a proof of the relationship by mathematical induction.

This kind of approach, whether we call it term-by-term, dynamic, iterative or incremental, seems to be relatively rare in school geometry, even though we have written about its over-use in 'investigational' work. Consider one of the most familiar school geometry theorems:

> **The interior angle sum of a triangle (in a plane) is 180°.**

Typically, this theorem is proved 'statically'. A 'single' triangle is drawn (a 'generic' triangle, ie one that is meant, at least implicitly, to represent 'all' triangles). The proof then involves drawing a line parallel to one of the sides and then using angle properties of parallel lines to show that the angles in the triangle are equivalent to angles on a straight line. Two possible approaches (*Construction A* and *Construction B*) are shown below.



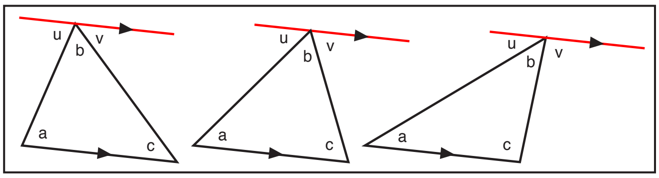
In *Construction A*, a line is drawn through a vertex, parallel to the opposite side.
So angle a = angle u, and angle c = angle v (alternate angles), and so $a + b + c = u + b + v = 180$ (angles on a straight line).

In *Construction B*, an exterior angle is first created at one of the vertices by extending one of the sides, and a line (or this time just a ray) is then drawn through that vertex, parallel to the opposite side.
So angle a = angle v, and angle b = angle w (corresponding and alternate angles), and so
$a + b + c = v + w + c = 180$ (angles on a straight line).

These are both well-known*, standard proofs, using basic geometrical properties that one might expect many secondary students to know - which suggests that many students may well be able to follow the proofs. (Though this still begs the question of how does one, or how did Euclid, hit upon such a proof?)

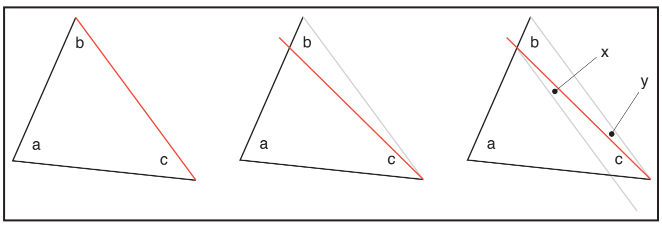
For some of the generalisation tasks discussed earlier (tile patterns, matchstick arrays, line-crossings, etc), we have argued that a static (generic) approach can often be more direct and illuminating than the more usual iterative (dynamic) approach. In the case of the current theorem (Interior angle sum of triangle), the classic (ie Euclidean) approach is static (generic), as it is for other familiar geometry theorems (eg the circle theorems). However, though this approach is logically coherent (and hence 'convincing'), it may not always give us (or our students) a good sense of *why* these familiar geometry theorems are true, eg why the angle sum is invariant for all triangles - a result which, if it weren't so familiar to us, might actually seem rather surprising!

To get a better sense of such invariance, it is worth supplementing the standard approach with a dynamic treatment. One obvious way of doing this is to drag a point, as illustrated by the sequence below for *Construction A* (this can be done using dynamic geometry software, or by means of sketches or in one's imagination). Although all the angles change in size,



the previous equalities still hold, ie it is still the case that $a = u$, $c = v$ and $a + b + c = u + b + c = 180°$.

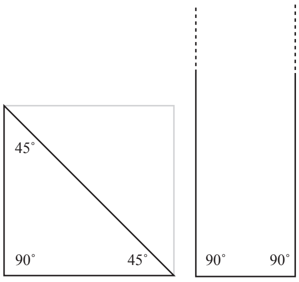
Another approach is to consider what happens when one of the sides is rotated slightly, as in the sequence below. Here it is fairly clear that as the angle b increases, the angle c decreases, but of course one still needs to show that the increase (x) exactly balances the decrease (y). (Note here that we are struggling with a common ambiguity in geometry, where the



* Some teachers will have been introduced to this kind of proof through the Key Stage 3 Framework document (DfEE, 2001). When Healy and Hoyles presented a set of 'proofs' of the theorem to a group of nearly 100 teachers in 1996, including a practical 'proof' of tearing off corners, 26 % chose the practical 'proof' as nearest to what they would do. With a similar sample of teachers in 2002, this went down to 10 %.

symbol "*b*", for example, is being used simultaneously to represent the name of an angle and its size - actually, it's worse here, as "*b*" is also being used to refer to a specific angle and to a family of angles!)

A variant of such a dynamic approach is to start with a triangle whose angles we 'know', so that we can establish that not only is the angle sum constant, it comes to two right angles (180°). The triangle might, for example, be formed by cutting a square along a diagonal (and hence with angles of 45°, 90° and 45°), or be one that is infinitely tall (and hence with angles 90°, 90° and 0°).



In this chapter we have looked at a variety of ways of tackling tasks involving generalisation and proof. We have argued that, in UK schools at least, there is an over-reliance on an empirical approach, where students generate data, look for patterns and then finally, perhaps, look for a structural explanation for their patterns. We have shown that often it is possible to look for structure much earlier on and we feel that teachers and students should be made more aware of this. However, we are not trying to say that the latter approach is always to be preferred. Exploring a task by generating examples might help students understand the task better - and a systematic, dynamic approach may be particularly helpful for developing a feel for ideas in geometry. Student-exploration might also lead to the discovery of interesting and unexpected patterns - and give students a greater sense of ownership and commitment to the task.

The effectiveness of a particular approach varies with the task being investigated and/or with the stage that the investigation has reached, eg with some tasks one might adopt a more empirical approach while one is forming conjectures, and a more structural approach as one tries to prove them.

These ideas are not new*. For example on page 154 of the Key Stage 3 Framework document for mathematics, we find this statement of attainment for sequences and functions. Here the notion of looking for structure is implicit in the reference to 'context'.

ALGEBRA

Pupils should be taught to:

Find the *n*th term, justifying its form by referring to the context in which it was generated

On the same page, there follows an example involving matchsticks, where both an empirical and a structural approach are presented. Thus it is suggested that students start by generating the first few arrangements of the matchstick pattern** and then find a pattern in the numbers generated (eg, the numbers 4, 7, 10, 13 go up in 3s). It is then suggested that this is explained ('justified') in terms of the context, ie the structure of the matchstick pattern ("Every square needs three matches, plus one more for the first square"). This is sound advice, as long as this search for a justification is not seen as just an afterthought.

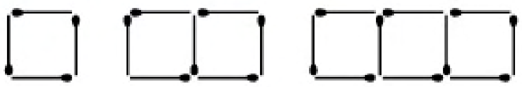
As outcomes, Year 7 pupils should, for example:

Generate sequences from simple practical contexts.

- Find the first few terms of the sequence.
- Describe how it continues by reference to the context.
- Begin to describe the general term, first using words, then symbols; justify the generalisation by referring to the context.

For example:

- Growing matchstick squares



Number of squares	1	2	3	4	...
Number of matchsticks	4	7	10	13	...

Justify the pattern by explaining that the first square needs 4 matches, then 3 matches for each additional square, or you need 3 matches for every square plus an extra one for the first square.

In the *n*th arrangement there are $3n+1$ matches.
Possible justification:
Every square needs three matches, plus one more for the first square, which needs four. For n squares, the number of matches needed is 3n, plus one for the first square.

* See Morgan, C (1998) *Writing Mathematically*, or Roper, T (1999) in A. Orton (Ed) *Pattern in the Teaching and Learning of Mathematics*, Chapter 12.

** This is perhaps fair enough as the matchstick pattern is presented as a 'growth pattern' and as the work is on sequences. However, *SMP 11-16* got round this by emphasising the existence of a rule between the number of matchsticks and squares that allowed one to generate any example of the pattern, rather than having to start at 'term 1', and by presenting individual arrays in a more or less random order. Of course, a term-by-term (incremental) approach can be useful in other mathematical areas, for example when one is looking at functions, and in particular at the gradients of their graphs.