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APPROACHES TO A CIRCLE-THEOREM TASK BY A NOVICE GROUP OF SECONDARY SCHOOL PUPILS

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A group of four Year 10 pupils were observed during a mathematics lesson as they attempted to solve a GCSE question involving circle theorems. The pupils were relatively inexperienced problem solvers and their knowledge of circle theorems was far from 'fluent'. This lack of experience cast a useful light on the challenges involved in solving geometric tasks which we as teachers might regard as quite routine.

INTRODUCTION

In an earlier paper (Küchemann and Hoyles, 2006), we described how a pair of Year 10 pupils went about solving two 'routine' circle theorem tasks. These pupils sometimes used powerful strategies - for example, making explicit the given property that a point was on a circle by drawing a radius from the point to the centre. However the pupils' use of the givens was not methodical, ie they had not yet fully developed the heuristic of 'What do I know and how can I use it?'. They also did not fully grasp the generic nature of geometric diagrams: thus they sometimes used perceptual reasoning which lead them to make claims about diagrams which were not necessarily true. The pupils benefited from working together but we also observed the importance of their having 'space' to think through their individual ideas.

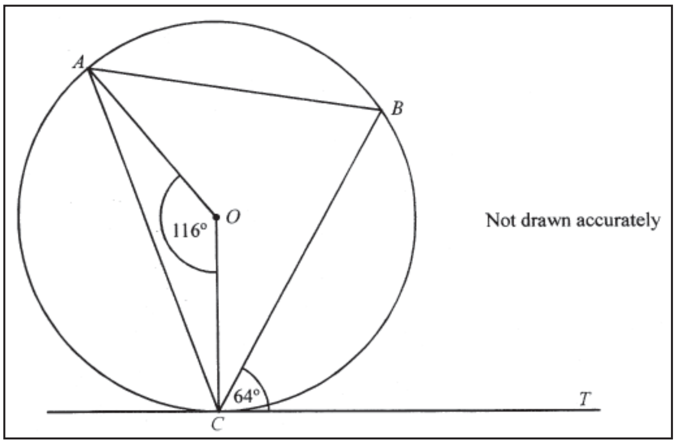
In this paper I look at a group of four Year 10 pupils, from a different school and a somewhat lower attaining mathematics set but using a similar GCSE geometry task. (The two schools were just a few miles apart and the teachers of the two sets of pupils had planned the work together, as part of their involvement in the *Proof Materials Project*.) I video-taped the group but also sat with them from time to time and sometimes asked them questions, usually in the form of a request for an explanation of what they were doing or thinking, but occasionally as an attempt to move them on when they seemed totally stuck.

The four pupils were used to sitting and working together, although they were not experienced problem solvers and their knowledge of the circle theorems was quite limited and far from fluent. Their teacher had gone over some of the theorems in the previous lesson and in the current lesson each group was provided with a reference set of stapled sheets describing various circle theorems. Each pupil also had a set of past GCSE examination questions that the teacher had compiled, edited and enlarged onto A3 paper. In each group the pupils were asked to choose one of the questions and to work through it together.

It might be thought that observing pupils working on a task about which they had little knowledge or experience would not be informative. In fact it brought into sharp relief some of the challenges that pupils face in tackling what might seem to us as teachers to be quite routine tasks.[1] These challenges included: **the use of diagrams** (and, for example, arguing from perception, and not appreciating the freedoms and constraints in a geometric diagram), **strategic thinking** (using the explicit and implicit givens, using known relationships and theorems, and working backwards to determine what it would be useful to know), and **systematising** (working in a mathematical domain that involves a network of relationships that are essentially two-way but on which we impose a direction and order).

THE TASK AND THE PUPILS' RESPONSES

The four pupils (who we shall call Alan, Anthony, Laura and Miles) decided to tackle part (a) of a question with the diagram shown below (O is the centre of the circle and TC is the tangent to the circle at C, and we are asked to "Write down the size of angle ABC"). [The teacher had not photocopied part (b) which asks pupils to prove that triangle ABC is isosceles.]



The group was rather diffident, perhaps because of my presence, but perhaps also because they were not particularly used to this kind of work which seemed rather strange to them. Thus they made slow progress. At the same time, there was a good cooperative atmosphere and they were prepared to listen to each others' ideas - though they did not always, or immediately, take an idea further, perhaps because they did not fully understand it or, as I had found with the pair of pupils mentioned above, because they were preoccupied with their own ideas.

I saw little sign of strategic thinking in the sense of deliberately trying to work towards the answer, or back from it, beyond the normal default strategy of 'what can we see?'. There was also little evidence of pupils asking 'How can we use the givens?'. Thus, although Anthony did say, at the very start of tackling the problem, "We have to do something with O...", no one made any reference to the fact that CT is a tangent and that this could be relevant, till much later. (In fact one does not need the tangent to answer part (a), though in the event the group did make use of it in their proof.)

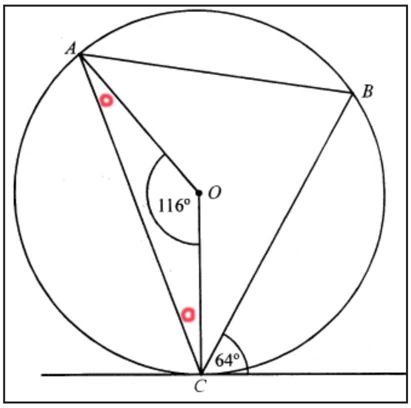
It is obviously intended that part (a) is solved by using the *Angle at the centre* theorem. However, because of their lack of familiarity with the circle theorems, none of the pupils in the group saw this. Instead it took them another 15 minutes to come up with a valid solution and one they were all reasonably happy with.

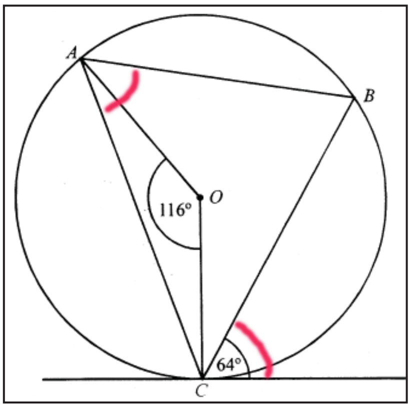
The first idea that surfaced was that triangle ABC might be isosceles, with CA = CB. This came from Alan, who acknowledged that he was just "Guessing them two sides are the same". Interestingly, Alan kept returning to this idea, where a more experienced pupil might have realised that this would probably be fruitless: there is no simple way of determining the lengths of the sides, so one would have to find equal angles to show the triangle is isosceles, in the course of which one would almost certainly find the desired angle ABC.

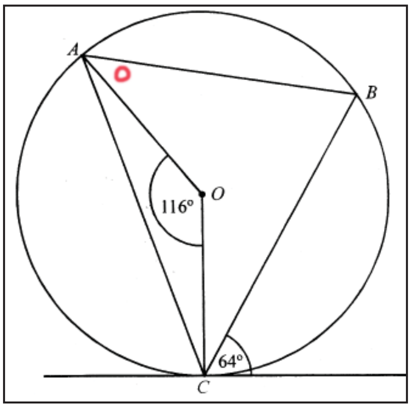
Shortly afterwards, Anthony had an insight which Miles picked up on, even though the conversation was quite cryptic:

Anthony	Ah yes, it’s a triangle... 180˚... that means these two are the same... but they’re divided.
Miles	So what’s 180 minus 116?
Alan	64...
Anthony	So that’s 32 and that’s 32.

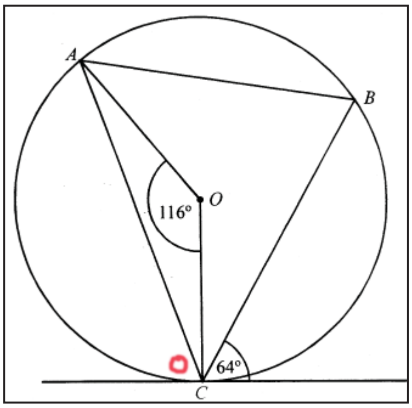
What they were discussing here was triangle AOC, which *is* isosceles (see the annotated diagram, below left). However, having found that the base angles are 32˚, it did not seem to get them any further. Miles briefly wondered whether the angle at O was twice the angle at A (which may have been an attempt to use *Angle at the centre*). He then started to go through the circle theorem reference-sheets and suddenly called out “Anthony, got it!” while pointing to the *Alternate segment* theorem. It still took him a while to apply the theorem to the given problem (below, middle), but he then went on to show that angle OAB was 64˚ – 32˚ = 32˚ (below, right).

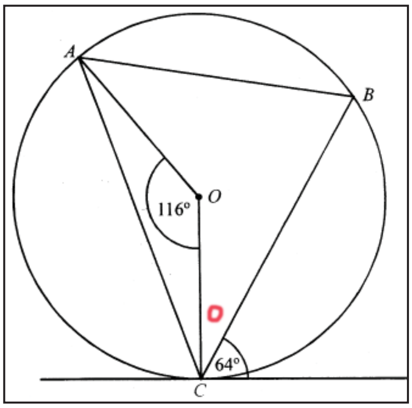




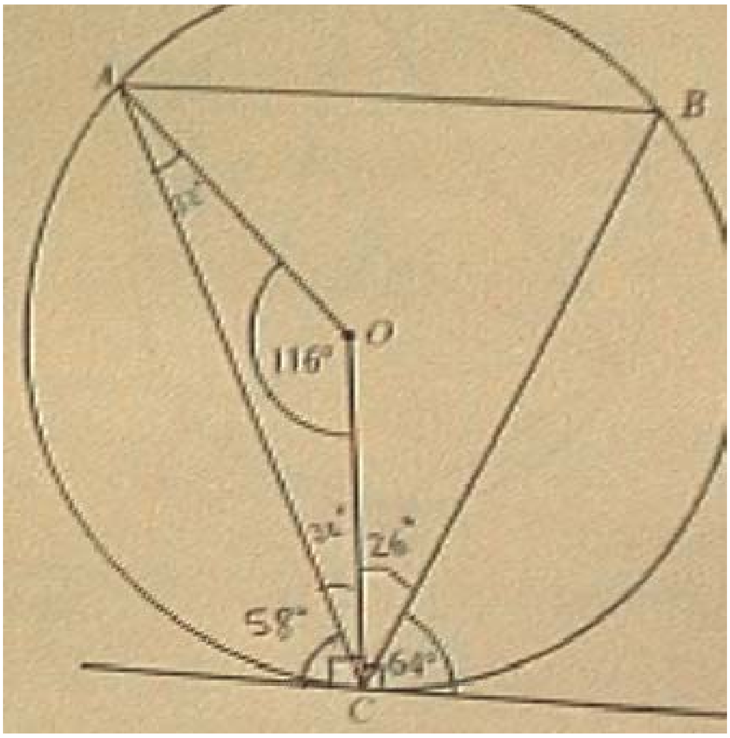


About a minute later (which seemed like an age), Anthony suddenly noticed the tangent: “Ah! That’s a right angle. Because of the tangent”. The group then decided to find the angle between the tangent and the chord AC (see below, middle), which they did by calculating 90 – 32 = 58. This would now have allowed them to find the desired angle at B, by applying the *Alternate segment* theorem again. However, no one saw this, which suggests there was no purposeful strategy behind their decision. Thus they would have been better off finding the unknown angle on the other side of the tangent, as this is inside the triangle ABC (far right) and





which could lead them to B by using the more familiar knowledge about the angle sum of triangles. Indeed, this is what they eventually did (by calculating 90 – 64 = 26). They then went on to find the whole angle C of the triangle ABC (by calculating 26 + 32 = 58).



At this stage, Anthony's diagram looked like this (right). It is interesting to note that Anthony had not yet marked the size of the total angle at A or the portion OAB, even though Miles had earlier established (or at least argued) that these were 64˚ and 32˚ respectively.
There followed a quite lengthy and intense discussion where the pupils seemed to be consolidating some of their earlier findings. Laura expressed uncertainty about the value of 32 for angle OAB which was shared by Anthony (perhaps because they were not sure about the *Alternate segment* theorem and how to apply it) and Anthony suggested "We've got to find out another way (to see) if it's 32". Alan returned to his earlier conjecture about the triangle ABC:

> I think that's an isosceles triangle, so that would be two equal angles and two equal sides. Then that (angle OAB) could be 26 (ie the same as OCB) or it could that one (angle ABC) that could be 58 (the same as angle ACB).

Laura went along with Alan's suggestion that OAB could be 26˚ but acknowledged that it was just a hunch. Alan then repeated the argument he had just made and at that point I decided to intervene:

> Did we not earlier say that this one (angle OAB) was 32˚, but it doesn't look it. Have you given up on that because it doesn't look it?

Anthony bit on this:

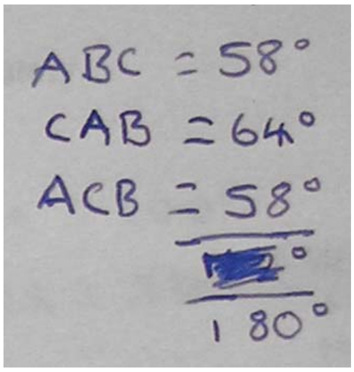
> I thought this angle (CAB) was corresponding to that one (TCB) ?

We thus established again (by means of the *Alternate segment* theorem, and a bit of prompting from me) that angle CAB was 64˚ (and that OAB was indeed 32˚). Anthony jumped in again:

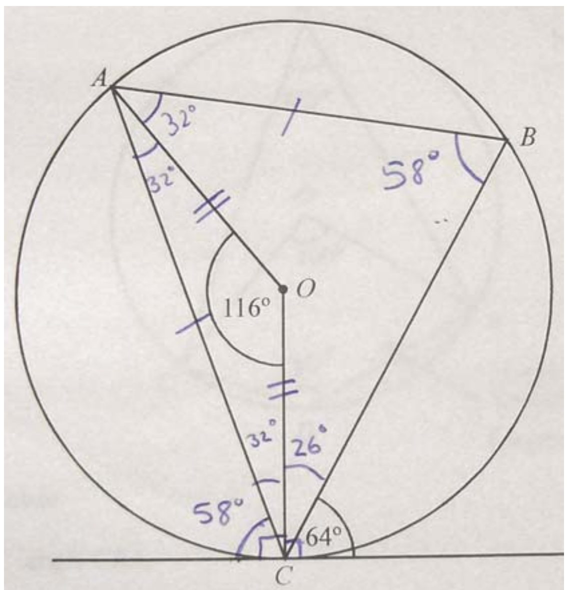
> So that's 58˚ [the desired angle ABC], because the three angles make 180.

Thus, after about 10 minutes of working on the task, one of the group had at last found a way of determining the angle ABC. Anthony explained that he had added 58 and 64 and that if you took this from 180 you got 58 for the angle. Miles seemed happy with this but Alan and Laura were not sure, particularly about the 64, which they seemed to think was based on the argument that the two angles at A are both 32, which of course is a reversal of Anthony's argument. Thus Anthony's use of the *Alternate segment* theorem (which he referred to by talking of 'corresponding angles') had not taken hold. When I point to the theorem on the reference-sheet Alan confessed that he had not really understand it when the teacher introduced it in the previous lesson.

To bring matters to a close, I suggested that we should accept the use of the theorem for now, and that therefore the desired angle ABC was indeed 58˚. Miles then noticed that triangle ABC had another angle of 58˚ and said, "If that is 58, that means the triangle is isosceles". Thus, Alan's hunch had been right after all, except that it turned out that AC and AB were the equal sides, not AC and BC. No one noticed that the angle between the tangent and the chord AC was also equal to angle ABC, so I pointed this out and asked whether this was just a coincidence. This prompted Miles to say "It's like that one and that one, the 64 and 64...", in other words another manifestation of the *Alternate segment* theorem.



While this closing conversation was going on, Anthony, with Laura's help, had been trying to sum the angles that the group had found for triangle ABC. He had some initial difficulty with the calculation, but now exclaimed, "Alan, I've got it! 180 in a triangle, so it's found it". It is not entirely clear from this statement whether Anthony was trying to check the arithmetic that had earlier been used to find angle ABC (ie 180 – angle A – angle C), or whether he was somehow trying to validate the steps used (or the result). However, his behaviour illustrates a common feature of this group's way of working, which we also noticed with the pair of pupils mentioned earlier. Namely that the pupils, though they were interested in and willing to engage with each other's ideas, were also frequently wrapped in their thoughts. It is tempting to imagine that with effective group work, pupils move forward on a common front. However, perhaps this group's behaviour gives a more accurate picture. Pupils bring different ideas and understandings to a problem so that as well as benefiting from each other's ideas they need space to develop and clarify their own. This also suggests that we should be careful not to ask pupils to work in groups too soon, ie they may sometimes benefit from thinking about a problem on their own, before taking on the ideas of others.



Anthony's completed script looked like this (right). By now the group had spent nearly 15 minutes discussion this one problem. However, they were pleased to have found a solution and called over the teacher to explain what they had done. This gave them a useful opportunity to rehearse their arguments, and the account they gave was generally clear and direct, though in their enthusiasm to explain, they talked over each other and so did not always stick to a common thread.

The group then decided to go on to another geometry question which also involved the *Angle at the centre* theorem. Anthony immediately spotted this:

> Ah, that's twice that one... That's where we went wrong on the other one... This is ... er... a ... I can't remember the names of them... that's the problem...

We see here again, the pupils' lack of familiarity with the circle theorems and the frustration this causes. Nonetheless, Anthony had recognised an appropriate theorem (even though he could not name it) and now realised that he could have used it in the previous question, which was an impressive breakthrough.

A notable feature of the whole lesson was that the teacher allowed pupils plenty of time to work on the questions they were given. The group of pupils that I observed benefited considerably from this and their solution to the first task was a substantial achievement which they were justly proud of. However, it has to be said they were severely hindered by their lack of familiarity with the circle theorems, which made it difficult for them to step back from the task and think about their problem-solving and proof strategies. Thus, though the circle theorems provide a potentially rich and challenging context for developing strategic thinking about proof, one has to think seriously about whether the effort required to achieve the necessary familiarity with the theorems will be worth it and whether one would be prepared to make available the necessary lesson time.

Acknowledgements

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NOTES

1. Of course, one can also learn a great deal from studying highly experienced and gifted mathematicians, of which the work of Krutetskii (1976) is a classic example.

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