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PUPILS' AWARENESS OF STRUCTURE ON TWO NUMBER/ALGEBRA QUESTIONS

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A large sample of high attaining pupils were given a written proof test in Yr 8 (age 13.5 years) and similar tests in Yrs 9 and 10. We look at their responses to two number/algebra questions which were designed to assess whether pupils used empirical or structural reasoning. We found that the use of structural reasoning increased over the years, albeit at a modest rate, but that the use of empirical reasoning, in the form of inappropriate number pattern spotting or through the desire to perform rather than analyse a calculation, was still widespread. However, we also identified a more advanced use of empirical reasoning, namely to check the validity of a structural argument.

The analysis presented here forms part of The Longitudinal Proof Project (Hoyles and Küchemann: http://www.ioe.ac.uk/proof), which looked at pupils' mathematical reasoning over time. Data were collected through annual surveying of high-attaining pupils from randomly selected schools within nine geographically diverse English regions. Pupils were given a written proof test at the end of Yr 8 (age 13.5 years) and given similar tests at the end of Yrs 9 and 10. Overall, 1512 pupils from 54 schools took all three tests. The tests comprised items in number/algebra and in geometry, some in open response format and some multiple choice. We discuss here two open response questions in number/algebra, A1 and A4.

There is a considerable body of research to suggest that school pupils tend to argue at an empirical level rather than on the basis of mathematical structure (Bell, 1976; Balacheff, 1988; Coe and Ruthven, 1994; Bills and Rowland, 1999). Of course, inductive reasoning can be important and fruitful in mathematics (Polya, 1954) but, in the UK at least, the emphasis on generating data and looking for number patterns, even at the upper end of secondary school, has often seemed to be at the detriment of looking for structure.

Item A1 involves a pattern of white tiles and grey tiles and was devised to see whether pupils could make a 'far generalisation' (Stacey, 1989) on the basis of the pattern's structure or whether they would resort to a 'function' strategy or 'whole-object scaling' (ibid) based on spurious number patterns. Brown et al (2002), in their work with student teachers, found that the students quite often chose to "perform computations when reasoning about computations would suffice". Question A4 explores this tendency. It is based on a suggestion made by Ruthven (1995) and concerns divisibility, but where the dividend is too large for computation to be a viable strategy.

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Pupils' responses to A1: changes in number pattern spotting from Yr 8 to Yr 10

Item A1 (which was familiar to English pupils) was asked in Yrs 8 and 10 (a different but parallel item was used in Yr 9). It is a standard number/algebra item involving a tile pattern, and was designed to test whether pupils could discern and describe a structure (assessed by carrying out and explaining a numerical calculation).

The Yr 8 version is shown in Fig. 1. We had deliberately built numerical distractors into the item, in the form of simple, but irrelevant, relationships between the numbers of white tiles mentioned in the item (namely, 6 × 10 = 60) and between the number of white and grey tiles in the given configuration (6 × 3 = 18).

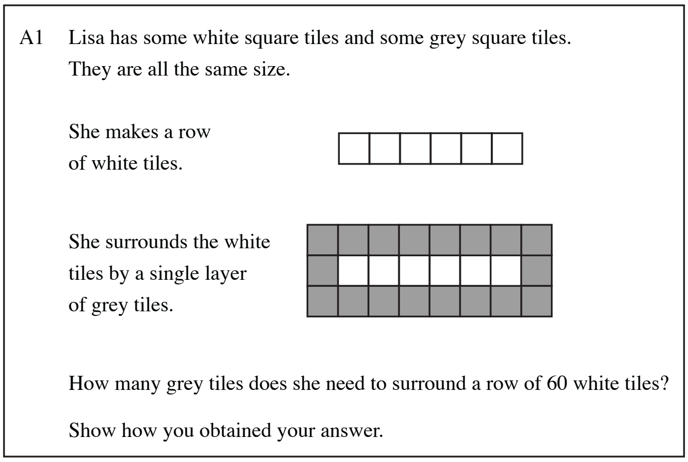

Figure 1: Item A1 (Yr 8 version)

Responses to question A1 were coded into 5 broad categories, or codes:

- Code 1: Spotting number patterns, no structure;
- Code 2: Some recognition of structure (incomplete or draws & counts);
- Code 3: Recognition and use of structure: specific (correct answer, eg showing 60+60+3+3);
- Code 4: Recognition and use of structure: general (correct answer and general rule, eg ×2, +6)
- Code 5: Recognition and use of structure: general, with use of variables (correct answer and general rule, with naming of variables in words or letters).

Analysis of the code frequencies revealed that a substantial minority of pupils had fallen for our distractors, in that they had used inappropriate 'number pattern spotting' strategies (code 1), such as 'There are 10 times as many white tiles so there will be 10 times as many grey tiles' (scaling approach), or 'There are 3 times as many grey tiles as white tiles' (function approach). The scaling approach was far more popular than the function approach, but both give the result 180 (10 × 18 or 3 × 60), and altogether 35% of the total sample gave such responses in Yr 8. This fell to 21% in Yr 9 but stayed at 21% in Yr 10.

Complementing the changes in frequency of pattern spotting responses, the frequency of correct responses (codes 3, 4 and 5) went up from 47% in Yr 8 to 68% in Yr 9 but only to 70% in Yr 10. However, this small increase from Yr 9 to Yr 10 masks a substantial rise in the use of variables (expressed in words or with letters) in pupils' explanations (code 5 responses), from 16% in Yr 9 to 26% in Yr 10 (starting from just 9% in Yr 8).

Longitudinal data showed that the improvement in pupils' responses from Yr 8 to 10 was far from smooth. Regarding the number pattern spotting responses in particular, while only about half of the pupils who gave such a response in Yr 8 or Yr 9, did so again the following year, we also found that of those who gave such a response in Yr

9 or Yr 10, about half had *not* done so the previous year, ie their responses had ‘regressed’. This suggests, that for some pupils at least, there is an element of chance about their responses: rather than being wedded to a particular way of construing such tile patterns (with some going for the underlying structure and some for a superficial number pattern) they seemed to hit upon one way on one occasion and another way on another occasion. An examination of individual scripts also shows that some pupils flipped between approaches on a given occasion. On the other hand, while 80% of those who gave a correct structural response in Yr 8 (codes 3, 4 or 5) also did so in Yr 10, only 54% of those giving a pattern spotting response in Yr 8 gave a correct structural response in Yr 10.

The use of algebra

The Yr 9 and 10 versions of question A1 had an added part, A1b, where students are asked to express the relationship between the numbers of tiles in algebra. It was worded as follows:

A1b: Write an expression for the number of grey tiles needed to surround a row of *n* white tiles.

We were interested in whether students were able to express any relationship they discerned in the tiling pattern in algebra, and indeed whether this was consistent with explanations of structure given in words or numbers. We have noted in a previous study (Healy & Hoyles, 2000) that Yr 10 students rarely used algebra as a language with which to described mathematical structure, even though they accorded high status to algebraic ‘proofs’.

A1b asks pupils to map the number of white tiles onto grey, ie it requires a function approach ($n \rightarrow 2n + 6$). Thus we were curious to see whether this would force a rethink on pupils who had used a scaling number pattern approach (18 grey tiles ×10 = 180 grey tiles). In the event, this turned out to be the case for over half these pupils. Thus for example, of the 197 pupils who gave a scaling number pattern response to A1 in Yr 10, 43% gave the response ‘$3n$’ (or equivalent) to A1b, ie they switched to a function response which still fitted their (incorrect) answer of 180 but no longer fitted the method they had used to obtain it. A further 13% of these pupils gave the correct response of ‘$2n + 6$’ (or equivalent) to A1b, ie abandoned their pattern spotting approach entirely. On the other hand, and not surprisingly, few of the 110 pupils giving a function number pattern response to A1 switched for A1b: 87% gave the response ‘$3n$’, which fits their pattern spotting answer and approach, with only 1% switching to a correct response. Interestingly too, 93% of those who had given a correct, but specific and non-algebraic response (code 3) to A1 in Y10 (which is all that the item requires), gave a correct algebraic response to A1b. We found similar response patterns in Yr 9.

As part of our research, we undertook case studies of some of the project schools (in fact those schools that had performed particularly well in the proof tests), which enabled us to interview some individual pupils after they had completed some or all of the tests, so we could ask them to reflect on the way they had answered in different years. We consider here three such pupils, MS, EC and JG, each of whom had given a number pattern spotting response to A1 in Yr 8 and/or Yr 10 (and sometimes also in Yr 9).

Pattern spotters

MS gave a pattern spotting response in Yr 8 and Yr 10 (and also in Yr 9); however in Y8 he used the '×10' strategy while in Yr 10 he used '×3' (his Yr 10 response to A1b, '3*n*', was consistent with this). He was interviewed a few days after taking the Yr 10 test so had little difficulty remembering his Yr 10 response:

MS: ...there's 6 there and 18…altogether times it by 3, then I thought it would be the same if you wanted to find out how many grey tiles would be in 60 so I timesed by 3…

He also seemed to have little difficulty in 'making sense' again of his Yr 8 response but seemed unperturbed by the fact that it was different (albeit leading to the same answer 180). His approach seems to have been fairly spontaneous and unreflective in both years.

It so happened we had also interviewed MS a year previously, a day after he had taken the Yr 9 Test. In this extract the interviewer asks him about his Yr 8 strategy of multiplying by 10, and again, there is no sign of any doubt in the validity of the method used:

I: Ok, and what makes you feel that that's the right way to do it?
MS: Because I found it the easiest way to get to 60 white tiles.
I: It's a nice quick way to get from 6 to 60 but, umm, how do you know it's the right way to get from 6 to 60 ?
MS: I just know, the sum of 6 times 10 is 60.

Pupil EC gave a '×10' pattern spotting response in Yr 8 (and Yr 9) but gave a correct answer in Yr 10 based on the geometric structure. He was interviewed about a week after taking the Yr 10 test and asked to compare his Yr 8 and 10 responses.

I: Which one do you think is the right one?
EC: I think this one is.
I: The Year 8 one.
EC: Yeah, kind of the first instinct I had.
I: You go by instinct.
EC: Yeah, I think, sort of, in the majority of the time the first instinct is right, so, I think maybe that one looks right.

EC is then asked to explain his Yr 10 response in more detail, which he seems able to do quite well. However, this is not enough to change his mind about the relative merits of his Yr 8 and 10 responses:

I: ... So, you ended up here in Year 10 with this double thing and then add six.
EC: Yeah.
I: So, how did that come to you, I mean, why would you have done that?
EC: I think, because we needed 60 and there was six along each row, each of the white things, so that means 12, so I just thought that doubling it, and then there's three left over, so I just, plussed three on one, so, I'm not really sure.
I: Okay. I mean, that sounds sensible enough, so, the trouble is, we've still got these two different answers. So are you going to stick with your instinct, your Year 8 instinct?

EC: I think so, yes.

Now, it is just possible that EC's statement above that "I just thought that doubling it", is evidence of an *empirical* generalisation (of the fact that 12 is double 6), rather than of genuine insight into the structure of the tile pattern; also he seems to have difficulty keeping track of all the structural elements, in that he mentions one "three left over" but not the other. Nonetheless, having demonstrated that he can make some use of the geometric structure to find the correct number of tiles and that he can describe crucial features of the structure to the interviewer, it seems surprising that he was so ready to abandon this potentially insightful approach for the sake of simplicity. Perhaps at this stage, through a lack of experience or guidance, EC does not have the meta knowledge needed to classify his different responses in an appropriate way and to recognise their positive or negative qualities. This is partly borne out by his responses to A1b, which are correct in Yr 9 and Yr 10, even though in Yr 9 he gave a scaling ('×10') pattern spotting response to the first part of A1.

JG had a similar set of responses to EC, in that she gave a '×10' number pattern response in Yr 8 and a structural response in Yr 10 ("60 times 2 is 120, 3 times 2 is 6, 120 plus 6 is 126"). She was interviewed about a week after taking the Yr 10 test. At first she could make no sense of her Yr 8 response ("I have no idea why I wrote that in Yr 8"), though she comes up with an interpretation eventually:

I: I mean, say someone else had done it, not you…could you sort of try and figure out why on earth they did it?
JG: (Long pause) No.
I: No, you can't see any logic in it?
JG: Well…yeah. I can now. It's because there's 6 there so, I figured 6 times 10 would be the 60 that they were talking about in the question, and so I just had to times the amount around the outside by the same number. Oh yeah.

However, unlike EC, she prefers her structural Yr 10 response which to her makes more sense:

I: Can you say a bit more why it makes more sense?
JG: I don't know…just a couple more years' practice of finding patterns and stuff.
I: So how does this answer sort of fit the pattern, the Year 10 answer fit the pattern better?
JG: Well because, I didn't just times the ones round the outside…by the same number as the ones on the inside…I worked out sort of a rule for it, rather than just a rule for that, that number.
I: Right. How did you get the rule for the Year 10 answer?
JG: Well, the three at each end won't change, it's a single row of tiles say…you just use the top…the grey tiles above and below the white…
I: Right, okay…

Her answers here are interesting. First she justifies her preference for the structural response with an external reason ("more years' practice"), which has nothing to do with the quality of the actual response and which is certainly no more valid than EC's quest for simplicity. However, she is able to describe the structure itself very nicely ("three at each end... ...grey tiles above and below...") and she does in fact say something, albeit in a rather cryptic way, about the quality of this explanation, namely about it being *general*: thus she found "sort of a rule" in contrast to "just a rule for that ... number". Notice also the statement about multiplying "the ones round the outside", which potentially provides a test for this number pattern spotting strategy, since the outcome would be a set of grey tiles that no longer fits snugly around the white tiles. Further, she gives correct responses to A1b (in Yr 9 and Yr 10) which shows she is able to express the structure using algebra. All in all, we seem here to be witnessing the beginnings of a meta knowledge about structural explanations, even though the concepts and/or language may not yet be well-formed.

We do not know how representative these few interviewees are, but their responses do suggest that, for some pupils at least, the simplicity of number pattern responses may have a stronger appeal than the insight that might be gained from taking a structural approach. It also points to possible discontinuities between modeling with numbers, narrative descriptions of these models and modeling with algebra. JG did have insight derived from a structural approach, though her attempts to describe the characteristics of her response were still quite limited. This is a phenomenon we have found elsewhere and which may well be widespread even amongst the highest of our high attaining pupils, since they will generally have had little experience of producing mathematical explanations and reflecting upon them.

Pupils' responses to A4: the tendency to calculate

We found a strong tendency for students to work at an empirical level on all our test items. In the case of A1, this was manifested by the pattern spotting responses discussed above, and by (the small number of) students who generated further data (e.g. for 7, 8 and 9 white tiles) and who used this to induce a rule. The same tendency was very strong in responses to question A4 (shown abbreviated in Fig. 2, below).

The question was used in Yrs 8 and 9 (but not in Yr 10). In part *a* students are asked about the divisibility of 5! by 3 and in part *c* about the divisibility of 100! by 31 (or 50! by 19).

Yr 8 version of A4	Yr 9 version of A4
A4 a) 4! means $4 \times 3 \times 2 \times 1$. 5! means $5 \times 4 \times 3 \times 2 \times 1$. Is 5! exactly divisible by 3 ? Explain your answer.	A4 a) 4! means $4 \times 3 \times 2 \times 1$. 5! means $5 \times 4 \times 3 \times 2 \times 1$. Is 5! exactly divisible by 3 ? Explain your answer.
b) What does 100! mean?	b) What does 50! mean?
c) Is 100! exactly divisible by 31 ? Explain your answer.	c) Is 50! exactly divisible by 19 ? Explain your answer.

Figure 2: Yr 8 and Yr 9 versions of question A4

Most students could correctly determine the divisibility of 5! by 3 (76% in Yr 8, 83% in Yr 9), but the overwhelming majority did so by evaluating 5! and by performing the calculation $120 \div 3$, with only 2% of the sample in Yr 8 and only 6% in Yr 9 basing their argument on the notion that 3 is a *given* factor of 5!. Students were not allowed calculators, so that the latter kind of argument was essential to answer part c) correctly, and in the event 3% of the sample did so in Yr 8, and 9% in Yr 9. Most students wanted to evaluate the factorial, and had no viable alternative strategy. Some students wrote statements like "That would take years to work out and if there is some short cut I don't know it". Some evaluated 100! as being 2400, on the basis that 100 = 20 × 5 so 100! = 20 × 5!, and so answered 'No'; others answered 'No' because 100! is even and 31 is odd or a prime.

We initially interpreted our interviews as suggesting that students *needed* to calculate and were insecure about using number relationships even when they apparently understood them. This would perhaps helps explain the seemingly astonishing fact that nearly half (26 of 55) of the students who did answer part *c* successfully in Yr 8 'regressed' in Yr 9. However, we have modified our views on this: Jahnke (2005) uses the metaphor of 'theoretical physicist' to describe pupils' behaviour as they learn to engage in mathematical proof. From this perspective, pupils' recourse to empirical evidence can be seen as a perfectly rational attempt to test the validity of a proof argument, rather than as a rejection or lack of appreciation of such arguments.

We illustrate this propensity to calculate on A4, by looking at our interviews with two students, MH and AM. From their written responses (summarised in Table 1, below) it appears that both students made considerable progress from Yr 8 to Yr 9, and both would seem to have a clear understanding of the 'divisibility principle' by Yr 9, as evidenced by the responses to part *c*. However, our interviews with them, which took place the day after they had taken the Yr 9 test, suggest their understanding is not quite so secure.

Yr 8	**Yr 9**
Pupil MH A4a: "Yes"; calculates 120 ÷ 3	**Pupil MH A4a**: "Yes"; calculates 120 ÷ 3
Pupil MH A4c: Attempts various calculations, eg of 10!, but gives up. Answers "No"	**Pupil MH A4c**: "Yes. If you times it by a certain number you will be able to divide by it"
Pupil AM A4a: "Yes"; calculates 120 ÷ 3	**Pupil AM A4a**: "Yes. The number has been multiplied by 3, so it must be divisible by 3"
Pupil AM A4c: Leaves blank	**Pupil AM A4c**: "Yes. The number has been multiplied by 19, so it must be divisible by 19"

Table 4: Pupil MH's and AM's responses to A4a and A4c in Yr 8 and Yr 9

In our interview with MH, he is first asked to say a bit more about his Yr 9 explanation to A4c. He replies with this rather nice way of describing the inverse relationship between multiplying and dividing by 19:

MH: Well if you times in from 50 down to 1 you times it by 19 somewhere, so if you want to divide it by 19 you'd just get the same answer as if not timesing it by 19 and just leaving it out altogether.

The interviewer then asks MH about the fact that he had changed his Yr 9 answer to part c) from No to Yes:

I: How did it hit you because interestingly you put 'no' first and then changed it to 'yes'. Was that a quick change, did you quickly see it or did you sit there and think?

MH: Sitting there and think about it 'cos then I figured out that it was divisible by 19 if you can times it by 19, so I changed it.

I: Did you, did it help, did you look back at part *a* ?

MH: Yes.

I: You did?

MH: Yes, I noticed that part *a*, everything you times in by it you can divide by, can be divided by 5, 4, 3, 2 and 1.
I: Now when you say you noticed that, is that because you know your tables well enough to know that 5 goes into 120 and four goes into it, or did you do it in your head and actually worked it out?
MH: In my head and worked it out.
I: You did, well did you work out every one?
MH: Yes.
I: What, you divided 120 by all these numbers?
MH: Yes.
I: Really, that's quite a lot, a lot to do. So do you think, are you then arguing that because it works for 5 you reckon it ought to work for 50 -
MH: - Yeh, it should do.
I: So your feeling is that all the numbers you multiply by you can divide by. Is that simply because it happens to work here, or is there some deeper reason for it, is it kind of...?
MH: I just feel that it works, and you know that it works in the first five, and when you get higher and higher and you times by any of the numbers that you can divide it by, and say in the 50-exclamation-mark you can divide something like 99, you're not going to be sure it can work or not, 'cept if you use calculator, which we wasn't allowed to.

This sequence, and especially the last sentence, might suggest that MH's awareness and acceptance of the inverse relationship between multiplication and division, is not entirely secure, and is based on an empirical generalisation (of the divisibility of 5! by 5, 4, 3, 2 and 1) rather than on genuine insight. However this is a harsh interpretation since to determine that, say, 50! is divisible by 19 one also needs to realise that the terms following ×19 in 50! (ie ×18, ×17, etc) do not affect the divisiblity.

AM's set of written responses is similar to that of MH, except that in Yr 9 he gives an explanation based on divisibility in part a) as well as part c), rather than one based on evaluating 5!. Thus, in part a) he had written "The number has been multiplied by 3, so it must be divisible by 3". In the interview, he is asked how he arrived at this explanation:

AM: I think I was just thinking about it. I was thinking 5! would be 5 × 4 × 3 × 2 ×1 so if it's been timesed by three you can almost certainly divide it by 3 and I was just thinking sort of that was that. Also I was saying because it's only then been timesed by 2 and by 1, because 1's obviously not going to change and 2's just going to double it. So you're just going to be able to divide it by 3.

This is interesting on several counts. First, though AM does not talk of calculating 5! his explanation is still very much grounded in the steps of the potential calculation. It is not enough that × 3 is one element in a string of factors, he carefully checks that the subsequent factors, × 2 and × 1, won't affect the divisibility by 3. This also nicely illustrates the fact that a notion like 'divisibility' involves a whole nexus of ideas (see Brown et al, 2002), including some understanding of the associative and commutative properties of multiplication. Last, AM says he is "almost certain" but not entirely that 5! is divisible by 3. The interviewer explores this further:

I: You said just now it's almost certainly divisible by 3, you're hesitating slightly.
AM: I think it is divisible by 3, I think at the time I wasn't completely certain.
I: What yesterday, but you're certain today?
AM: Yes, after all that I'm almost certain that's good.
I: Why is there still this edge of doubt? You say *almost* certain.

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AM: Don’t know. The thing is, I am certain, but not quite... I can’t see why, I can see slightly why it works but not entirely, I haven’t thought ‘suppose you had a bigger number, would it…’

I: You have with part *c* made the same conclusion because you said the number’s been multiplied by 19 so it must be divisible by 19, you’ve got the word ‘must’ interestingly, not ‘it might be’, ‘almost certainly will’, so you do seem to think that a bigger number would work. So say we had a thousand factorial and we wanted to know whether 128 went into it?

AM: Yeh, it would be divisible by 128... as far as I know (*laughs*), but it’s very hard to be certain.

Thus it would seem that for AM and for MH too, their lack of certainty about the ‘divisibility principle’ (the argument that if you multiply by a certain number the result is divisible by that number) is not because they do not understand the basic argument or appreciate its power, but because of some awareness that other features of the situation (eg that in 100!, the term × 31 is followed by a long string of other terms) may render the principle invalid.

Conclusion

The propensity to calculate occurred on many of our items. For example, in Yr 10 we asked pupils to prove the statement that “When you add any 2 odd numbers, your answer is always even”. The most common approach was a purely empirical one (given by 31% of the sample), whereby pupils simply demonstrated that the statement was true for one of more pair of odd numbers. This was followed in popularity by an exhaustive approach (19% of the sample) concerned with the units digits of odd an even numbers. This can lead to a valid proof but is still essentially empirical in that it is concerned with surface features of odd and even numbers rather than underlying structure. On the other hand, a substantial minority did attempt proofs containing some element of structure, expressed in narrative (18%) visual (6%) or algebraic (5%) form.

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The tendency to adopt an empirical approach is very strong in English schools and beyond, which might at least partly be explained by particular classroom approaches (see for example Morgan, 1997). Our view is that the situation does not have to be like this – we can change pupils’ habits of mind (Cuoco et al, 1996). Our findings lends some support to this, since our pupils did make some, albeit modest, progress in the use of structural reasoning, and the data suggest that switching strategies (even between incorrect strategies) might be helpful in catalyzing a new perspective on a problem. In our current work (Proof Materials Project) we are working with teachers to see whether this change of habit can be put into effect more widely, in particular by making pupils more aware of different kinds of proof strategies and explanations.

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References

Balacheff, N.: 1988, 'Aspects of proof in pupils' practice of school mathematics', in D. Pimm (ed.), *Mathematics, Teachers and Children*, Hodder and Stoughton, London, pp. 216-235.

Bell, A. W.: 1976, 'A study of pupils' proof-explanations in mathematical situations', *Educational Studies in Mathematics* 7, 23-40.

Bills, L. and Rowland, T.: 1999, 'Examples, generalisation and proof', in L. Brown (ed.), *Making Meanings in Mathematics*. Refereed Proceedings of the British Society for Research into Learning Mathematics, QED Publications, York, pp. 103-116.

Brown, A., Thomas, K. and Tolias, G.: 2002, 'Conceptions of divisibility: success and understanding', in S. R. Campbell and R. Zazkis (eds.), *Learning and Teaching Number Theory*, Ablex Publishing, Connecticut, pp. 41-82.

Coe, R., and Ruthven, K.: 1994, 'Proof practices and constructs of advanced mathematics students', *British Educational Research Journal* 20(1), 41-53.

Cuoco, A., Goldenberg, E. P. and Mark, J.: 1996, 'Habits of Mind: an organizing principle for mathematics curricula', *J. Mathematical Behavior* 15, 375 - 402.

Healy, L. and Hoyles, C.: 2000, 'A study of proof conceptions in algebra', *Journal for Research in Mathematics Education*, 31 (4), 396-428.

Jahnke, H. N.: 2005, 'A genetic approach to proof'. Paper presented at the *Fourth Congress of the European Society for Research in Mathematics Education* (CERME 4), Februrary 2005, Sant Feliu de Guixols, Spain.

Morgan, C.: 1997, 'The institutionalisation of open-ended investigation: some lessons from the UK experience', in E. Pehkonen (ed.), *Use of Open-Ended Problems in Mathematics Classroom*, Department of Teacher Education, Helsinki, Research Report, 176, 49-62.

Polya, G.: 1954, *Mathematics and Plausible Reasoning, Volume 1: Induction and Analogy in Mathematics*, OUP, London.

Ruthven, K: 1995, 'The unbearable mightness of being', in L Healy (ed), *Justifying and Proving in Mathematics: Draft Activities*. Unpublished manuscript, Institute of Education, London.

Stacey, K.: 1989, 'Finding and using patterns in linear generalising problems', *Educational Studies in Mathematics* 20, 147-164.